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II YEAR

FINANCIAL MATHEMATICS

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SMAS41: FINANCIAL MATHEMATICS

SYLLABUS

UNIT-I:

Probability and Normal Random Variables

Chapter 1: Sections: 1.1-1.8

UNIT-II:

Brownian Motion and Geometric Brownian Motion

Chapter 2: Sections: 2.1-2.3

UNIT-III:

Interest Rate and Present Value Analysis

Chapter 3: Sections: 3.1-3.4

UNIT-IV:

Pricing Contracts via Arbitrage

Chapter 4: Sections: 4.1-4.2

UNIT-V:

The Arbitrage Theorem

Chapter 5: Sections: 5.1-5.3

Recommended Text

Sheldon M. Ross, An Introduction to Mathematical Finance: Options and Other Topics, Second Edition, Cambridge University Press, First published 2002.

Reference Book

I. Karatzas and S.E.Shreve, Methods of Mathematical Finance, Springer, 1998.

SMAS41: FINANCIAL MATHEMATICS

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UNIT-I:

Probability and Normal Random Variables

Chapter 1: Sections: 1.1-1.8

1. Probability

1.1. Probabilities and Events:

Consider an experiment and let S, called the sample space, be the set of all possible outcomes of the experiment. If there are m possible outcomes of the experiment then we will generally number them 1 through m, and so $S = \{1,2, \dots, m\}$. However, when dealing with specific examples, we will usually give more descriptive names to the outcomes.

Example 1.1.1:

(i) Let the experiment consist of flipping a coin, and let the outcome be the side that lands face up. Thus, the sample space of this experiment is $S = \{h, t\}$,

where the outcome is h if the coin shows heads and t if it shows tails.

(ii) If the experiment consists of rolling a pair of dice - with the outcome being the pair (i, j), where i is the value that appears on the first die and j the value on the second - then the sample space consists of the following 36 outcomes:

$$\begin{aligned}
&(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),\\
&(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),\\
&(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),\\
&(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),\\
&(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),\\
&(6,1), (6,2), (6,3), (6,4), (6,5), (6,6).
\end{aligned}$$

(iii) If the experiment consists of a race of r horses numbered $1,2,3. \dots, r$, and the outcome is the order of finish of these horses, then the sample space is

$S = \{$ all orderings of the numbers $1,2,3, \ldots, r\}$.

For instance, if $r = 4$ then the outcome is $(1,4,2,3)$ if the number-1 horse comes in first, number 4 comes in second, number 2 comes in third, and number 3 comes in fourth.

Consider once again an experiment with the sample space $S = \{1,2, \ldots, m\}$. We will now suppose that there are numbers $p_1, \ldots, p_m$ with

$$p_i \geq 0, i = 1, \ldots, m, \text{ and } \sum_{i=1}^{m} p_i = 1$$

and such that p_i is the probability that i is the outcome of the experiment.

Example 1.1.2:

In Example 1.1.1(i), the coin is said to be fair or unbiased if it is equally likely to land on heads as on tails. Thus, for a fair coin we would have that

$$p_h = p_t = 1/2.$$

If the coin were biased and heads were twice as likely to appear as tails, then we would have

$$p_h = 2/3, p_t = 1/3.$$

If an unbiased pair of dice were rolled in Example 1.1.1(ii), then all possible outcomes would be equally likely and so

$$p_{(i.j)} = 1/36, 1 \leq i \leq 6, 1 \leq j \leq 6.$$

If $r = 3$ in Example 1.1.1 (iii), then we suppose that we are given the six nonnegative numbers that sum to 1 :

$$p_{1,2,3}, p_{1,3,2}, p_{2,1,3}, p_{2,3,1}, p_{3,1,2}, p_{3,2,1}$$

where $p_{i,j,k}$ represents the probability that horse i comes in first, horse j second, and horse k third.

Any set of possible outcomes of the experiment is called an event. That is, an event is a subset of S, the set of all possible outcomes. For any event A, we say that A occurs whenever the outcome of the experiment is a point in A. If we let $P(A)$ denote the probability that event A occurs, then we can determine it by using the equation

$P(A) = \sum_{i \in A} p_i$ .... (1.1)

Note that this implies $P(S) = \sum_i p_i = 1$ ....(1.2)

In words, the probability that the outcome of the experiment is in the sample space is equal to 1 - which, since S consists of all possible outcomes of the experiment, is the desired result.

Example 1.1.3:

Suppose the experiment consists of rolling a pair of fair dice. If A is the event that the sum of the dice is equal to 7 , then $A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

and $P(A) = 6/36 = 1/6$

If we let B be the event that the sum is 8 , then

$P(B) = p_{(2,6)} + p_{(3,5)} + p_{(4,4)} + p_{(5,3)} + p_{(6,2)} = 5/36.$

If, in a horse race between three horses, we let A denote the event that horse number 1 wins, then $A = \{(1,2,3),(1,3,2)\}$ and

$P(A) = p_{1,2,3} + p_{1,3,2}$

For any event A, we let A^c, called the complement of A, be the event containing all those outcomes in S that are not in A. That is, A^c occurs if and only if A does not. Since

$$1 = \sum_{i} p_i$$
$$= \sum_{i \in A} p_i + \sum_{i \in A^c} p_i$$
$$= P(A) + P(A^c),$$

we see that $P(A^c) = 1 - P(A)$ ………..(1.3)

That is, the probability that the outcome is not in A is 1 minus the probability that it is in A. The complement of the sample space S is the null event $\emptyset$, which contains no outcomes.

Since $\emptyset = S^c$, we obtain from Equations (1.2) and (1.3) that $P(\emptyset) = 0$.

For any events A and B we define $A \cup B$, called the union of A and B, as the event consisting of all outcomes that are in A, or in B, or in both A and B. Also, we define their intersection AB (sometimes written $A \cap B$) as the event consisting of all outcomes that are both in A and in B.

Example 1.1.4:

Let the experiment consist of rolling a pair of dice. If A is the event that the sum is 10 and B is the event that both dice land on even numbers greater than 3 , then

$A = \{(4,6), (5,5), (6,4)\}, B = \{(4,4), (4,6), (6,4), (6,6)\}.$

Therefore,

$$A \cup B = \{(4,4), (4,6), (5,5), (6,4), (6,6)\},$$
$$AB = \{(4,6), (6,4)\}.$$

For any events A and B, we can write

$$P(A \cup B) = \sum_{i \in A \cup B} p_i,$$
$$P(A) = \sum_{i \in A} p_i,$$
$$P(B) = \sum_{i \in B} p_i.$$

Since every outcome in both A and B is counted twice in $P(A) + P(B)$ and only once in $P(A \cup B)$, we obtain the following result, often called the addition theorem of probability.

Proposition 1.1.1.

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

Thus, the probability that the outcome of the experiment is either in A or in B is: the probability that it is in A, plus the probability that it is in B, minus the probability that it is in both A and B.

Example 1.1.5:

Suppose the probabilities that the Dow-Jones stock index increases today is 0.54, that it increases tomorrow is 0.54, and that it increases both days is 0.28. What is the probability that it does not increase on either day?

Solution:

Let A be the event that the index increases today, and let B be the event that it increases tomorrow. Then the probability that it increases on at least one of these days is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= 0.54 + 0.54 - 0.28 = 0.80 \end{aligned}$$

Therefore, the probability that it increases on neither day is $1 - 0.80 = 0.20$.

If $AB = \emptyset$, we say that A and B are mutually exclusive or disjoint. That is, events are mutually exclusive if they cannot both occur. Since $P(\emptyset) = 0$, it follows from Proposition 1.1.1 that, when A and B are mutually exclusive,

$$P(A \cup B) = P(A) + P(B)$$

1.2. Conditional Probability:

Suppose that each of two teams is to produce an item, and that the two items produced will be rated as either acceptable or unacceptable. The sample space of this experiment will then consist of the following four outcomes:

$$S = \{(a, a), (a, u), (u, a), (u, u)\},$$

where (a, u) means, for instance, that the first team produced an acceptable item and the second team an unacceptable one. Suppose that the probabilities of these outcomes are as follows:

$$P(a, a) = 0.54,$$
$$P(a, u) = 0.28,$$
$$P(u, a) = 0.14,$$
$$P(u, u) = 0.04.$$

If we are given the information that exactly one of the items produced was acceptable, what is the probability that it was the one produced by the first team? To determine this probability, consider the following reasoning. Given that there was exactly one acceptable item produced, it follows that the outcome of the experiment was either (a, u) or (u, a). Since the outcome (a, u) was initially twice as likely as the outcome (u, a), it should remain twice as likely given the information that one of them occurred. Therefore, the probability that the outcome was (a, u) is $2/3$,

whereas the probability that it was (u, a) is $1/3$.

Let $A = \{(a, u), (a, a)\}$ denote the event that the item produced by the first team is acceptable, and let $B = \{(a, u), (u, a)\}$ be the event that exactly one of the produced items is acceptable. The probability that the item produced by the first team was acceptable given that exactly one of the produced items was acceptable is called the conditional probability of A given that B has occurred; this is denoted as $P(A \mid B)$.

A general formula for $P(A \mid B)$ is obtained by an argument similar to the one given in the preceding. Namely, if the event B occurs then, in order for the event A to occur, it is necessary that the occurrence be a point in both A and B; that is, it must be in AB. Now, since we know that B has occurred, it follows that B can be thought of as the new sample space, and hence the probability that the event AB occurs will equal the probability of AB relative to the probability of B. That is,

$P(A \mid B) = \frac{P(AB)}{P(B)}$ …………(1.4)

Example 1.2.1:

A coin is flipped twice. Assuming that all four points in the sample space

$S = \{(h,h),(h,t),(t,h),(t,t)\}$ are equally likely, what is the conditional probability that both flips land on heads, given that
(a) the first flip lands on heads, and
(b) at least one of the flips lands on heads?

Solution.

Let $A = \{(h,h)\}$ be the event that both flips land on heads; let $B = \{(h,h),(h,t)\}$ be the event that the first flip lands on heads; and let $C = \{(h,h),(h,t),(t,h)\}$ be the event that at least one of the flips lands on heads. We have the following solutions:
and

$$\begin{aligned} P(A \mid C) &= \frac{P(AC)}{P(C)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t),(t,h)\})} \\ &= \frac{1/4}{3/4} \\ &= 1/3 \end{aligned}$$

Many people are initially surprised that the answers to parts (a) and (b) are not identical. To understand why the answers are different, note first that - conditional on the first flip landing on heads - the second one is still equally likely to land on either heads or tails, and so the probability in part (a) is $1/2$. On the other hand, knowing that at least one of the flips lands on heads is equivalent to knowing that the outcome is not (t, t). Thus, given that at least one of the flips lands on heads, there remain three equally likely possibilities, namely $(h, h), (h, t), (t, h)$, showing that the answer to part (b) is $1/3$.

It follows from Equation (1.4) that $P(AB) = P(B)P(A \mid B)$ ………..(1.5)

That is, the probability that both A and B occur is the probability that B occurs multiplied by the conditional probability that A occurs given that B occurred; this result is often called the multiplication theorem of probability.

Example 1.2.2:

Suppose that two balls are to be withdrawn, without replacement, from an urn that contains 9 blue and 7 yellow balls. If each ball drawn is equally likely to be any of the balls in the urn at the time, what is the probability that both balls are blue?

Solution:

Let B_1 and B_2 denote, respectively, the events that the first and second balls withdrawn are blue. Now, given that the first ball withdrawn is blue, the second ball is equally likely to be any of the remaining 15 balls, of which 8 are blue. Therefore, $P(B_2 \mid B_1) = 8/15$. As $P(B_1) = 9/16$, we see that

$$P(B_1B_2) = \frac{9}{16}\frac{8}{15} = \frac{3}{10}$$

The conditional probability of A given that B has occurred is not generally equal to the unconditional probability of A. In other words, knowing that the outcome of the experment is an element of B generally changes the probability that it is an element of A. (What if A and B are mutually exclusive?) In the special case where $P(A \mid B)$ is equal to $P(A)$, we say that A is independent of B. Since

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

we see that A is independent of B if $P(AB) = P(A)P(B)$ …………..(1.6)

The relation in (1.6) is symmetric in A and B. Thus it follows that, whenever A is independent of B, B is also independent of A - that is, A and B are independent events.

Example 1.2.3:

Suppose that, with probability 0.52 the closing price of a stock is at least as high as the close on the previous day, and that the results for successive days are independent. Find the probability that the closing price goes down in each of the next four days, but not on the following day.

Solution:

Let A_i be the event that the closing price goes down on day i. Then, by independence, we have

$$P(A_1A_2A_3A_4A_5^c) = P(A_1)P(A_2)P(A_3)P(A_4)P(A_5^c)$$
$$= (0.48)^4(0.52) = .0276.$$

1.3 Random Variables and Expected Values:

Numerical quantities whose values are determined by the outcome of the experiment are known as random variables. For instance, the sum obtained when rolling dice, or the number of heads that result in a series of coin flips, are random variables. Since the value of a random variable is determined by the outcome of the experiment, we can assign probabilities to each of its possible values.

Example 1.3.1:

Let the random variable X denote the sum when a pair of fair dice are rolled. The possible values of X are $2,3,\dots,12$, and they have the following probabilities:

$$P\{X=2\}=P\{(1,1)\}=1/36,$$
$$P\{X=3\}=P\{(1,2),(2,1)\}=2/36,$$
$$P\{X=4\}=P\{(1,3),(2,2),(3,1)\}=3/36,$$
$$P\{X=5\}=P\{(1,4),(2,3),(3,2),(4,1)\}=4/36,$$
$$P\{X=6\}=P\{(1,5),(2,4),(3,3),(4,2),(5,1)\}=5/36,$$
$$P\{X=7\}=P\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}=6/36,$$
$$P\{X=8\}=P\{(2,6),(3,5),(4,4),(5,3),(6,2)\}=5/36,$$
$$P\{X=9\}=P\{(3,6),(4,5),(5,4),(6,3)\}=4/36,$$
$$P\{X=10\}=P\{(4,6),(5,5),(6,4)\}=3/36,$$
$$P\{X=11\}=P\{(5,6),(6,5)\}=2/36,$$
$$P\{X=12\}=P\{(6,6)\}=1/36.$$

If X is a random variable whose possible values are $x_1, x_2, \dots, x_n$, then the set of probabilities $P\{X = x_j\}(j = 1, \dots, n)$ is called the probability distribution of the random variable. Since X must assume one of these values, it follows that

$$\sum_{j=1}^{n} P\{X = x_j\} = 1.$$

Definition:

If X is a random variable whose possible values are $x_1, x_2, \dots, x_n$, then the expected value of X, denoted by $E[X]$, is defined by $E[X] = \sum_{j=1}^{n} x_j P\{X = x_j\}$.

Alternative names for $E[X]$ are the expectation or the mean of X. In words, $E[X]$ is a weighted average of the possible values of X, where the weight given to a value is equal to the probability that X assumes that value.

Example 1.3.2:

Let the random variable X denote the amount that we win when we make a certain bet. Find $E[X]$ if there is a 60% chance that we lose 1 , a 20% chance that we win 1 , and a 20% chance that we win 2.

Solution:

$E[X] = -1(0.6) + 1(0.2) + 2(0.2) = 0$

Thus, the expected amount that is won on this bet is equal to 0. A bet whose expected winnings is equal to 0 is called a fair bet.

Example 1.3.3:

A random variable X, which is equal to 1 with probability p and to 0 with probability $1 - p$, is said to be a Bernoulli random variable with parameter p. Its expected value is

$E[X] = 1(p) + 0(1 - p) = p$

A useful and easily established result is that, for constants a and b,

$E[aX + b] = aE[X] + b$ …………. (1.7)

To verify Equation (1.7), let $Y = aX + b$. Since Y will equal $ax_j + b$ when $X = x_j$, it follows that

$$\begin{aligned} E[Y] &= \sum_{j=1}^{n} (ax_j + b)P\{X = x_j\} \\ &= \sum_{j=1}^{n} ax_j P\{X = x_j\} + \sum_{j=1}^{n} bP\{X = x_j\} \\ &= a\sum_{j=1}^{n} x_j P\{X = x_j\} + b\sum_{j=1}^{n} P\{X = x_j\} \\ &= aE[X] + b. \end{aligned}$$

An important result is that the expected value of a sum of random variables is equal to the sum of their expected values.

Proposition 1.3.1:

For random variables $X_1, \dots, X_k$,

$$E\left[\sum_{j=1}^{k} X_j\right] = \sum_{j=1}^{k} E[X_j].$$

Example 1.3.4:

Consider n independent trials, each of which is a success with probability p. The random variable X, equal to the total number of successes that occur, is called a binomial random variable with parameters n and p. We can determine its expectation by using the representation $X = \sum_{j=1}^{n} X_j$

where X_j is defined to equal 1 if trial j is a success and to equal 0 otherwise. Using Proposition 1, we obtain that $E[X] = \sum_{j=1}^{n} E[X_j] = np$

where the final equality used the result of Example 1.3.3.

The random variables $X_1, \dots, X_n$ are said to be independent if probabilities concerning any subset of them are unchanged by information as to the values of the others.

Example 1.3.5:

Suppose that k balls are to be randomly chosen from a set of N balls, of which n are red. If we let X_i equal 1 if the i th ball chosen is red and 0 if it is black, then $X_1, \dots, X_n$ would be independent if each selected ball is replaced before the next selection is made but they would not be independent if each selection is made without replacing previously selected balls. (Why not?)

Whereas the average of the possible values of X is indicated by its expected value, its spread is measured by its variance.

Definition:

The variance of X, denoted by $\text{Var}(X)$, is defined by $\text{Var}(X) = E[(X - E[X])^2]$

In other words, the variance measures the average square of the difference between X and its expected value.

Example 1.3.6:

Find $\text{Var}(X)$ when X is a Bernoulli random variable with parameter p.

Solution. Because $E[X] = p$ (as shown in Example 1.3 c), we see that

$$(X - E[X])^2 = \begin{cases} (1-p)^2 & \text{with probability } p \\ p^2 & \text{with probability } 1-p \end{cases}$$

Hence,

$$\begin{aligned} \text{Var}(X) &= E[(X - E[X])^2] \\ &= (1-p)^2 p + p^2(1-p) \\ &= p - p^2 \end{aligned}$$

If a and b are constants, then

$$\begin{aligned} \text{Var}(aX + b) &= E[(aX + b - E[aX + b])^2] \\ &= E[(aX - aE[X])^2] \quad \text{(by equation 1.7)} \\ &= E[a^2(X - E[X])^2] \\ &= a^2\, Var(X) \quad \ldots\ldots\ldots\ldots\ldots (1.8) \end{aligned}$$

Although it is not generally true that the variance of the sum of random variables is equal to the sum of their variances, this is the case when the random variables are independent.

Proposition 1.3.2:

If $X_1, \ldots, X_k$ are independent random variables, then $\text{Var}\left(\sum_{j=1}^{k} X_j\right) = \sum_{j=1}^{k} \text{Var}(X_j)$

Example 1.3.7:

Find the variance of X, a binomial random variable with parameters n and p.

Solution:

Recalling that X represents the number of successes in n independent trials (each of which is a success with probability p), we can represent it as $X = \sum_{j=1}^{n} X_j$

where X_j is defined to equal 1 if trial j is a success and 0 otherwise. Hence,

$$\begin{aligned} \text{Var}(X) &= \sum_{j=1}^{n} \text{Var}(X_j) \quad \text{(by Proposition 2)} \\ &= \sum_{j=1}^{n} p(1-p) \quad \text{(by Example 6)} \\ &= np(1-p) \end{aligned}$$

The square root of the variance is called the standard deviation. As we shall see, a random variable tends to lie within a few standard deviations of its expected value.

1.4. Covariance and Correlation:

The covariance of any two random variables X and Y, denoted by $\text{Cov}(X, Y)$, is defined by

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])].$$

Upon multiplying the terms within the expectation, and then taking expectation term by term, it can be shown that

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

A positive value of the covariance indicates that X and Y both tend to be large at the same time, whereas a negative value indicates that when one is large the other tends to be small. (Independent random variables have covariance equal to 0)

Example 1.4.1:

Let X and Y both be Bernoulli random variables. That is, each takes on either the value 0 or 1. Using the identity $\operatorname{Cov}(X,Y) = E[XY] - E[X]E[Y]$

and noting that XY will equal 1 or 0 depending upon whether both X and Y are equal to 1 , we obtain that

$$\operatorname{Cov}(X,Y) = P\{X = 1, Y = 1\} - P\{X = 1\}P\{Y = 1\}$$

From this, we see that

$$\begin{aligned}\operatorname{Cov}(X,Y) > 0 &\Leftrightarrow P\{X = 1, Y = 1\} > P\{X = 1\}P\{Y = 1\} \\ &\Leftrightarrow \frac{P\{X = 1, Y = 1\}}{P\{X = 1\}} > P\{Y = 1\} \\ &\Leftrightarrow P\{Y = 1 \mid X = 1\} > P\{Y = 1\}\end{aligned}$$

That is, the covariance of X and Y is positive if the outcome that $X = 1$ makes it more likely that $Y = 1$ (which, as is easily seen, also implies the reverse).

The following properties of covariance are easily established. For random variables X and Y, and constant c :

$$\begin{aligned}\operatorname{Cov}(X,Y) &= \operatorname{Cov}(Y,X) \\ \operatorname{Cov}(X,X) &= \operatorname{Var}(X) \\ \operatorname{Cov}(cX,Y) &= c\operatorname{Cov}(X,Y), \\ \operatorname{Cov}(c,Y) &= 0\end{aligned}$$

Covariance, like expected value, satisfies a linearity property - namely,

$$\operatorname{Cov}(X_1 + X_2, Y) = \operatorname{Cov}(X_1, Y) + \operatorname{Cov}(X_2, Y) \qquad \ldots\ldots\ldots\ldots\ldots (1.9)$$

Equation (1.9) is proven as follows:

$$\begin{aligned}\text{Cov}(X_1 + X_2, Y) &= E[(X_1 + X_2)Y] - E[X_1 + X_2]E[Y] \\ &= E[X_1Y + X_2Y] - (E[X_1] + E[X_2])E[Y] \\ &= E[X_1Y] - E[X_1]E[Y] + E[X_2Y] - E[X_2]E[Y] \\ &= \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)\end{aligned}$$

Equation (1.9) is easily generalized to yield the following useful identity:

$$\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \text{Cov}(X_i, Y_j) \quad \ldots\ldots\ldots (1.10)$$

Equation (1.10) yields a useful formula for the variance of the sum of random variables:

$$\begin{aligned}\text{Var}\left(\sum_{i=1}^{n} X_i\right) &= \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} \text{Cov}(X_i, X_i) + \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(X_i, X_j) \quad \ldots\ldots\ldots\ldots (1.11)\end{aligned}$$

The degree to which large values of X tend to be associated with large values of Y is measured by the correlation between X and Y, denoted as $\rho(X, Y)$ and defined by

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

It can be shown that

$$-1 \leq \rho(X, Y) \leq 1$$

If X and Y are linearly related by the equation $Y = a + bX$

then $\rho(X, Y)$ will equal 1 when b is positive and -1 when b is negative.

Exercises 1:

1.When typing a report, a certain typist makes i errors with probability $p_i (i \geq 0)$, where

$p_0 = 0.20, p_1 = 0.35, p_2 = 0.25, p_3 = 0.15$

What is the probability that the typist makes
(a) at least four errors;
(b) at most two errors?

2. A family picnic scheduled for tomorrow will be postponed if it is either cloudy or rainy. If the probability that it will be cloudy is 0.40, the probability that it will be rainy is 0.30, and the probability that it will be both rainy and cloudy is 0.20, what is the probability that the picnic will not be postponed?

3. If two people are randomly chosen from a group of eight women and six men, what is the probability that
(a) both are women;
(b) both are men;
(c) one is a man and the other a woman?

4. A club has 120 members, of whom 35 play chess, 58 play bridge, and 27 play both chess and bridge. If a member of the club is randomly chosen, what is the conditional probability that she
(a) plays chess given that she plays bridge;
(b) plays bridge given that she plays chess?

5. Cystic fibrosis (CF) is a genetically caused disease. A child that receives a CF gene from each of its parents will develop the disease either as a teenager or before, and will not live to adulthood. A child that receives either zero or one CF gene will not develop the disease. If an individual has a CF gene, then each of his or her children will independently receive that gene with probability $1/2$.
(a) If both parents possess the CF gene, what is the probability that their child will develop cystic fibrosis?

(b) What is the probability that a 30 -year old who does not have cystic fibrosis, but whose sibling died of that disease, possesses a CF gene?

6. Two cards are randomly selected from a deck of 52 playing cards. What is the conditional probability they are both aces, given that they are of different suits?

7. If A and B are independent, show that so are

(a) A and B^c;

(b) A^c and B^c.

8. A gambling book recommends the following strategy for the game of roulette. It recommends that the gambler bet 1 on red. If red appears (which has probability $18/38$ of occurring) then the gambler should take his profit of 1 and quit. If the gambler loses this bet, he should then make a second bet of size 2 and then quit. Let X denote the gambler's winnings.

(a) Find $P\{X > 0\}$.

(b) Find $E[X]$.

9. Four buses carrying 152 students from the same school arrive at a football stadium. The buses carry (respectively) 39, 33, 46, and 34 students. One of the 152 students is randomly chosen. Let X denote the number of students who were on the bus of the selected student. One of the four bus drivers is also randomly chosen. Let Y be the number of students who were on that driver's bus.

(a) Which do you think is larger, $E[X]$ or $E[Y]$?

(b) Find $E[X]$ and $E[Y]$.

10. Two players play a tennis match, which ends when one of the players has won two sets. Suppose that each set is equally likely to be won by either player, and that the results from different sets are independent. Find (a) the expected value and (b) the variance of the number of sets played.

11.A lawyer must decide whether to charge a fixed fee of $\$5{,}000$ or take a contingency fee of $\$25{,}000$ if she wins the case (and 0 if she loses). She estimates that her probability of winning is .30. Determine the mean and standard deviation of her fee if

(a) she takes the fixed fee;

(b) she takes the contingency fee.

12. Let $X_1, \ldots, X_n$ be independent random variables, all having the same distribution with expected value μ and variance σ^2. The random variable $\bar{X}$, defined as the arithmetic average of these variables, is called the sample mean. That is, the sample mean is given by $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$

(a) Show that $E[\bar{X}] = \mu$.

(b) Show that $\operatorname{Var}(\bar{X}) = \sigma^2/n$.

The random variable S^2, defined by $S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$ is called the sample variance.

(c) Show that $\sum_{i=1}^{n} (X_i - \bar{X})^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$.

(d) Show that $E[S^2] = \sigma^2$.

13.Verify that $\operatorname{Cov}(X, Y) = E[XY] - E[X]E[Y]$.

Exercise 1.15 Prove:

(a) $\operatorname{Cov}(X, Y) = \operatorname{Cov}(Y, X)$;

(b) $\operatorname{Cov}(X, X) = \operatorname{Var}(X)$;

(c) $\operatorname{Cov}(cX, Y) = c\operatorname{Cov}(X, Y)$;

(d) $\operatorname{Cov}(c, Y) = 0$.

14. If U and V are independent random variables, both having variance 1 , find $\operatorname{Cov}(X, Y)$ when $X = aU + bV, Y = cU + dV$

15.If $\operatorname{Cov}\left(X_i, X_j\right) = ij$, find

(a) $\operatorname{Cov}(X_1 + X_2, X_3 + X_4)$;

(b) $\operatorname{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$.

Normal Random Variables

1.5. Continuous Random Variables:

Whereas the possible values of the random variables considered in the previous chapter constituted sets of discrete values, there exist random variables whose set of possible values is instead a continuous region. These continuous random variables can take on any value within some interval. For example, such random variables as the time it takes to complete an assignment, or the weight of a randomly chosen individual, are usually considered to be continuous.

Every continuous random variable X has a function f associated with it. This function, called the probability density function of X, determines the probabilities associated with X in the following manner. For any numbers $a < b$, the area under f between a and b is equal to the probability that X assumes a value between a and b. That is,

$P\{a \leq X \leq b\} =$ area under f between a and b.

Figure 1.1 presents a probability density function.

1.6. Normal Random Variables:

A very important type of continuous random variable is the normal random variable. The probability density function of a normal random variable X is determined by two parameters, denoted by μ and σ, and is given by the formula

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, -\infty < x < \infty$$

A plot of the normal probability density function gives a bell-shaped curve that is symmetric about the value μ, and with a variability that is measured by σ. The larger the value of σ, the more spread there is in f. Figure 1.2 presents three different normal probability density functions.

Note how the curve flattens out as σ increases.

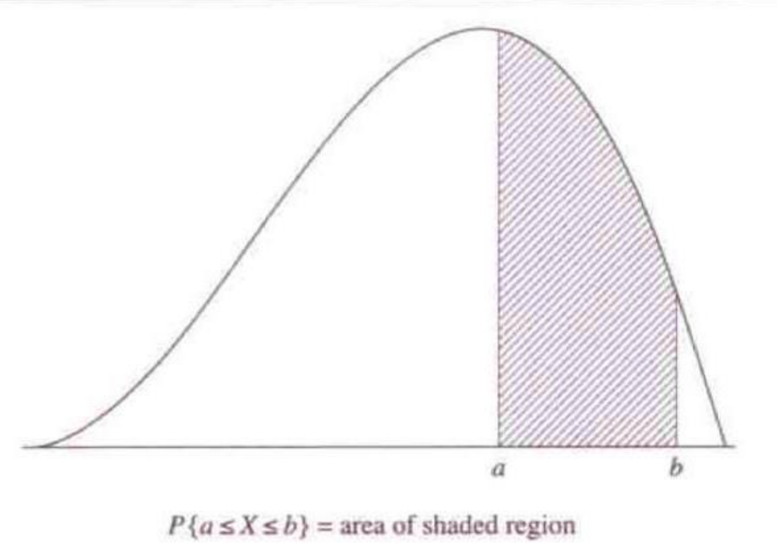

Figure 1.1. Probability Density Function of X

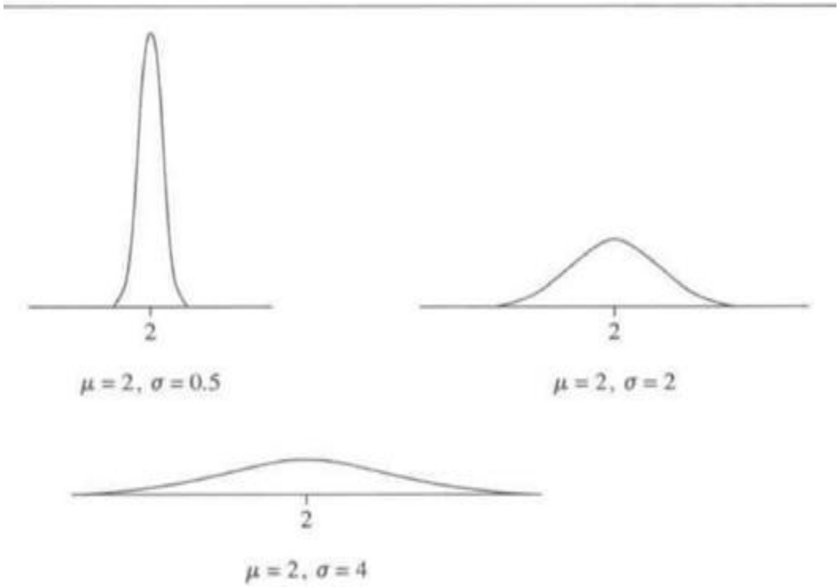

Figure 1.2. Three Normal Probability Density Function

It can be shown that the parameters μ and σ^2 are equal to the expected value and to the variance of X, respectively. That is,

$\mu = E[X], \sigma^2 = \text{Var}(X)$

A normal random variable having mean 0 and variance 1 is called a standard normal random variable. Let Z be a standard normal random variable. The function $\Phi(x)$, defined for all real numbers x by

$$\Phi(x) = P\{Z \leq x\}$$

is called the standard normal distribution function. Thus $\Phi(x)$, the probability that a standard normal random variable is less than or equal to x, is equal to the area under the standard normal density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < x < \infty$$

between $-\infty$ and x. Table 1.1 specifies values of $\Phi(x)$ when $x > 0$. Probabilities for negative x can be obtained by using the symmetry of the standard normal density about 0 to conclude (see Figure 1.3) that $P\{Z < -x\} = P\{Z > x\}$

or, equivalently, that $\Phi(-x) = 1 - \Phi(x)$

Example 1.6.1:

Let Z be a standard normal random variable. For $a < b$, express $P\{a < Z \leq b\}$ in terms of Φ.

Solution:

Since $P\{Z \leq b\} = P\{Z \leq a\} + P\{a < Z \leq b\}$

we see that

$$P\{a < Z \leq b\} = \Phi(b) - \Phi(a)$$

Example 1.6.2:

Tabulated values of $\Phi(x)$ show that, to four decimal places,

$P\{|Z| \leq 1\} = P\{-1 \leq Z \leq 1\} = 0.6826$
$P\{|Z| \leq 2\} = P\{-2 \leq Z \leq 2\} = 0.9544$
$P\{|Z| \leq 3\} = P\{-3 \leq Z \leq 3\} = 0.9974$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706

1.9	. 9713	. 9719	. 9726	. 9732	. 9738	. 9744	. 9750	. 9756	. 9761	. 9767
2.0	. 9772	. 9778	. 9783	. 9788	. 9793	. 9798	. 9803	. 9808	. 9812	. 9817
2.1	. 9821	. 9826	. 9830	. 9834	. 9838	. 9842	. 9846	. 9850	. 9854	. 9857
2.2	. 9861	. 9864	. 9868	. 9871	. 9875	. 9878	. 9881	. 9884	. 9887	. 9890
2.3	. 9893	. 9896	. 9898	. 9901	. 9904	. 9906	. 9909	. 9911	. 9913	. 9916
2.4	. 9918	. 9920	. 9922	. 9925	. 9927	. 9929	. 9931	. 9932	. 9934	. 9936
2.5	. 9938	. 9940	. 9941	. 9943	. 9945	. 9946	. 9948	. 9949	. 9951	. 9952
2.6	. 9953	. 9955	. 9956	. 9957	. 9959	. 9960	. 9961	. 9962	. 9963	. 9964
2.7	. 9965	. 9966	. 9967	. 9968	. 9969	. 9970	. 9971	. 9972	. 9973	. 9974
2.8	. 9974	. 9975	. 9976	. 9977	. 9977	. 9978	. 9979	. 9979	. 9980	. 9981
2.9	. 9981	. 9982	. 9982	. 9983	. 9984	. 9984	. 9985	. 9985	. 9986	. 9986
3.0	. 9987	. 9987	. 9987	. 9988	. 9988	. 9989	. 9989	. 9989	. 9990	. 9990
3.1	. 9990	. 9991	. 9991	. 9991	. 9992	. 9992	. 9992	. 9992	. 9993	. 9993
3.2	. 9993	. 9993	. 9994	. 9994	. 9994	. 9994	. 9994	. 9995	. 9995	. 9995
3.3	. 9995	. 9995	. 9995	. 9996	. 9996	. 9996	. 9996	. 9996	. 9996	. 9997
3.4	. 9997	. 9997	. 9997	. 9997	. 9997	. 9997	. 9997	. 9997	. 9997	. 9998

Table 1.1: $\boldsymbol{\Phi(x) = P\{Z \leq x\}}$

When greater accuracy than that provided by Table 1.1 is needed, the following approximation to $\Phi(x)$, accurate to six decimal places, can be used: For $x > 0$,

$$\Phi(x) \approx 1 - \frac{1}{\sqrt{2\pi}} e^{-x^2/2}(a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5),$$

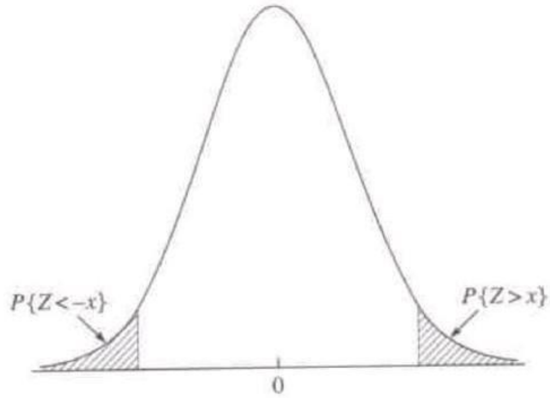

Figure 1.3: $\boldsymbol{P(Z < -x) = P(Z > x)}$

Where
$$\begin{aligned} y &= \frac{1}{1+.2316419x} \\ a_1 &= 0.319381530 \\ a_2 &= -0.356563782 \\ a_3 &= 1.781477937 \\ a_4 &= -1.821255978 \\ a_5 &= 1.330274429 \end{aligned}$$

And $\Phi(-x) = 1 - \Phi(x)$

1.7. Properties of Normal Random Variables:

An important property of normal random variables is that if X is a normal random variable then so is $aX + b$, when a and b are constants. This property enables us to transform any normal random variable X into a standard normal random variable. For suppose X is normal with mean μ and variance σ^2. Then, since (from Equations (1.7) and (1.8))

$$Z = \frac{X - \mu}{\sigma}$$

has expected value 0 and variance 1 , it follows that Z is a standard normal random variable. As a result, we can compute probabilities for any normal random variable in terms of the standard normal distribution function Φ.

Example 1.7.1:

IQ examination scores for sixth-graders are normally distributed with mean value 100 and standard deviation 14.2. What is the probability that a randomly chosen sixth-grader has an IQ score greater than 130?

Solution:

Let X be the score of a randomly chosen sixth-grader. Then,

$$P\{X > 130\} = P\left\{\frac{X-100}{14.2} > \frac{130-100}{14.2}\right\}$$
$$= P\left\{\frac{X-100}{14.2} > 2.113\right\}$$
$$= 1 - \Phi(2.113)$$
$$= 0.017$$

Example 1.7.2:

Let X be a normal random variable with mean μ and standard deviation σ. Then, since

$X - \mu \leq a\sigma$ is equivalent to $\frac{X-\mu}{\sigma} \leq a$

it follows from Example 1.6.2. that 68.26% of the time a normal random variable will be within one standard deviation of its mean; 95.44% of the time it will be within two standard deviations of its mean; and 99.74% of the time it will be within three standard deviations of its mean.

Another important property of normal random variables is that the sum of independent normal random variables is also a normal random variable. That is, if X_1 and X_2 are independent normal random variables with means μ_1 and μ_2 and with standard deviations σ_1 and σ_2, then $X_1 + X_2$ is normal with mean

$$E[X_1 + X_2] = E[X_1] + E[X_2] = \mu_1 + \mu_2$$

and variance

$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = \sigma_1^2 + \sigma_2^2$

Example 1.7.3:

The annual rainfall in Cleveland, Ohio, is normally distributed with mean 40.14 inches and standard deviation 8.7 inches. Find the probability that the sum of the next two years' rainfall exceeds 84 inches.

Solution:

Let X_i denote the rainfall in year $i(i = 1,2)$. Then, assuming that the rainfalls in successive years can be assumed to be independent, it follows that $X_1 + X_2$ is normal with mean 80.28 and variance $2(8.7)^2 = 151.38$. Therefore, with Z denoting a standard normal random variable,

$$\begin{aligned} P\{X_1 + X_2 > 84\} &= P\left\{Z > \frac{84 - 80.28}{\sqrt{151.38}}\right\} \\ &= P\{Z > .3023\} \\ &\approx .3812 \end{aligned}$$

The random variable Y is said to be a lognormal random variable with parameters μ and σ if $\log(Y)$ is a normal random variable with mean μ and variance σ^2. That is, Y is lognormal if it can be expressed as

$$Y = e^X$$

where X is a normal random variable. The mean and variance of a lognormal random variable are as follows:

$$E[Y] = e^{\mu+\sigma^2/2}$$
$$\text{Var}(Y) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

Example 1.7.4:

Starting at some fixed time, let $S(n)$ denote the price of a certain security at the end of n additional weeks, $n \geq 1$. A popular model for the evolution of these prices assumes that the price ratios $S(n)/S(n-1)$ for $n \geq 1$ are independent and identically distributed (i.i.d.) lognormal random variables. Assuming this model, with lognormal parameters $\mu = .0165$ and $\sigma = .0730$, what is

the probability that

(a) the price of the security increases over each of the next two weeks;

(b) the price at the end of two weeks is higher than it is today?

Solution:

Let Z be a standard normal random variable. To solve part (a), we use that $\log(x)$ increases in x to conclude that $x > 1$ if and only if $\log(x) > \log(1) = 0$. As a result, we have

$$\begin{aligned} P\left\{\frac{S(1)}{S(0)} > 1\right\} &= P\left\{\log\left(\frac{S(1)}{S(0)}\right) > 0\right\} \\ &= P\left\{Z > \frac{-.0165}{.0730}\right\} \\ &= P\{Z > -.2260\} \\ &= P\{Z < .2260\} \\ &\approx .5894 \end{aligned}$$

Therefore, the probability that the price is up after one week is 0.5894. Since the successive price ratios are independent, the probability that the price increases over each of the next two weeks is $(.5894)^2 = 0.3474$. To solve part (b), reason as follows:

$$\begin{aligned} P\left\{\frac{S(2)}{S(0)} > 1\right\} &= P\left\{\frac{S(2)}{S(1)}\frac{S(1)}{S(0)} > 1\right\} \\ &= P\left\{\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right) > 0\right\} \\ &= P\left\{Z > \frac{-.0330}{.0730\sqrt{2}}\right\} \\ &= P\{Z > -.31965\} \\ &= P\{Z < .31965\} \\ &\approx 0.6254 \end{aligned}$$

where we have used that $\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right)$, being the sum of independent normal random variables with a common mean 0.0165 and a common standard deviation 0.0730 , is itself a normal random variable with mean 0.0330 and variance $2(.0730)^2$.

1.8. The Central Limit Theorem:

The ubiquity of normal random variables is explained by the central limit theorem, probably the most important theoretical result in probability.

This theorem states that the sum of a large number of independent random variables, all having the same probability distribution, will itself be approximately a normal random variable.
For a more precise statement of the central limit theorem, suppose that $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables, each with expected value μ and variance σ^2, and let

$$S_n = \sum_{i=1}^{n} X_i.$$

Central Limit Theorem For large n, S_n will approximately be a normal random variable with expected value $n\mu$ and variance $n\sigma^2$. As a result, for any x we have

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} \approx \Phi(x)$$

with the approximation becoming exact as n becomes larger and larger. Suppose that X is a binomial random variable with parameters n and p. Since X represents the number of successes in n independent trials, each of which is a success with probability p, it can be expressed as

$$X = \sum_{i=1}^{n} X_i,$$

where X_i is 1 if trial i is a success and is 0 otherwise. Since (from Section 1.3)

$E[X_i] = p$ and $\text{Var}(X_i) = p(1-p)$

it follows from the central limit theorem that, when n is large, X will approximately have a normal distribution with mean np and variance $np(1-p)$.

Example 1.8.1:

A fair coin is tossed 100 times. What is the probability that heads appear fewer than 40 times?

Solution:

If X denotes the number of heads, then X is a binomial random variable with parameters $n = 100$ and $p = 1/2$. Since $np = 50$ we have $np(1-p) = 25$, and so

$$\begin{aligned} P\{X < 40\} &= P\left\{\frac{X-50}{\sqrt{25}} < \frac{40-50}{\sqrt{25}}\right\} \\ &= P\left\{\frac{X-50}{\sqrt{25}} < -2\right\} \\ &\approx \Phi(-2) \\ &= .0228. \end{aligned}$$

A computer program for computing binomial probabilities gives the exact solution .0176, and so the preceding is not quite as accurate as we might like. However, we could improve the approximation by noting that, since X is an integral-valued random variable, the event that $X < 40$ is equivalent to the event that $X < 39 + c$ for any $c, 0 < c \leq 1$. Consequently, a better approximation may be obtained by writing the desired probability as $P\{X < 39.5\}$. This gives

$$\begin{aligned} P\{X < 39.5\} &= P\left\{\frac{X-50}{\sqrt{25}} < \frac{39.5-50}{\sqrt{25}}\right\} \\ &= P\left\{\frac{X-50}{\sqrt{25}} < -2.1\right\} \\ &\approx \Phi(-2.1) \\ &= .0179 \end{aligned}$$

which is indeed a better approximation.

Exercises 2:

1.For a standard normal random variable Z, find:

(a) $P\{Z < -.66\}$;

(b) $P\{|Z| < 1.64\}$:

(c) $P\{|Z| > 2.20\}$.

2.Find the value of x when Z is a standard normal random variable and

$P(-2 < Z < -1\} = P\{1 < Z < x\}$.

3.Argue (a picture is acceptable) that $P\{|Z| > x\} = 2P\{Z > x\}$,

where $x > 0$ and Z is a standard normal random variable.

4.Let X be a normal random variable having expected value μ and variance σ^2, and let $Y = a + bX$. Find values $a, b(a \neq 0\ )$ that give Y the same distribution as X. Then, using these values, find $\mathrm{Cov}(X, Y)$.

5.The systolic blood pressure of male adults is normally distributed with a mean of 127.7 and a standard deviation of 19.2.

(a) Specify an interval in which the blood pressures of approximately 68% of the adult male population fall.

(b) Specify an interval in which the blood pressures of approximately 95% of the adult male population fall.

(c) Specify an interval in which the blood pressures of approximately 99.7% of the adult male population fall.

6.Suppose that the amount of time that a certain battery functions is a normal random variable with mean 400 hours and standard deviation 50 hours. Suppose that an individual owns two such batteries, one of which is to be used as a spare to replace the other when it fails. (a) What is the probability that the total life of the batteries will exceed 760 hours?

(b) What is the probability that the second battery will outlive the first by at least 25 hours?
(c) What is the probability that the longer-lasting battery will outlive the other by at least 25 hours?

7.The time it takes to develop a photographic print is a random variable with mean 18 seconds and standard deviation 1 second. Approximate the probability that the total amount of time that it takes to process 100 prints is

(a) more than 1,710 seconds;

(b) between 1,690 and 1,710 seconds.

8.Frequent fliers of a certain airline fly a random number of miles each year, having mean and standard deviation of 25,000 and 12,000 miles, respectively. If 30 such people are randomly chosen, approximate the probability that the average of their mileages for this year will

(a) exceed 25,000 ;

(b) be between 23,000 and 27,000 .

9. A model for the movement of a stock supposes that, if the present price of the stock is s, then - after one time period - it will either be us with probability p or ds with probability $1 - p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30% after the next 1,000 time periods if $u = 1.012, d = .990$, and $p = .52$.

10.In each time period, a certain stock either goes down 1 with probability .39 , remains the same with probability .20 , or goes up 1 with probability .41 . Assuming that the changes in successive time periods are independent, approximate the probability that, after 700 time periods, the stock will be up more than 10 from where it started.

Unit II

Brownian Motion and Geometric Brownian Motion

Chapter 2: Sections: 2.1-2.3

2. Geometric Brownian Motion

2.1. Geometric Brownian Motion:

Suppose that we are interested in the price of some security as it evolves over time. Let the present time be time 0 , and let $S(y)$ denote the price of the security a time y from the present. We say that the collection of prices $S(y), 0 \leq y < \infty$, follows a geometric Brownian motion with drift parameter μ and volatility parameter σ if, for all nonnegative values of y and t, the random variable

$\frac{S(t+y)}{S(y)}$ is independent of all prices up to time y; and if, in addition, $\log\left(\frac{S(t+y)}{S(y)}\right)$

is a normal random variable with mean μt and variance $t\sigma^2$.In other words, the series of prices will be a geometric Brownian motion if the ratio of the price a time t in the future to the present price will, independent of the past history of prices, have a log normal probability distribution with parameters μt and $t\sigma^2$.

It follows that a consequence of assuming a security's prices follow a geometric Brownian motion is that, once μ and σ are determined, it is only the present price - and not the history of past prices - that affects probabilities of future prices. Furthermore, probabilities concerning the ratio of the price a time t in the future to the present price will not depend on the present price. (Thus, for instance, the model implies that the probability a given security doubles in price in the next month is the same no matter whether its present price is 10 or 25.)

It turns out that, for a given initial price $S(0)$, the expected value of the price at time t depends on both of the geometric Brownian motion parameters. Specifically, if the initial price is s_0, then

$$E[S(t)] = s_0 e^{t(\mu+\sigma^2/2)}$$

Thus, under geometric Brownian motion, the expected price grows at the rate $\mu + \sigma^2/2$.

2.2 Geometric Brownian Motion as a Limit of Simpler Models:

Let Δ denote a small increment of time and suppose that, every Δ time units, the price of a security either goes up by the factor u with probability p or goes down by the factor d with probability $1 - p$, where

$$u = e^{\sigma\sqrt{\Delta}}, d = e^{-\sigma\sqrt{\Delta}},$$
$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right).$$

That is, we are supposing that the price of the security changes only at times that are integral multiples of Δ; at these times, it either goes up by the factor u or down by the factor d. As we take Δ smaller and smaller, so that the price changes occur more and more frequently (though by factors that become closer and closer to 1), the collection of prices becomes a geometric Brownian motion. Consequently, geometric Brownian motion can be approximated by a relatively simple process, one that goes either up or down by fixed factors at regularly specified times.

Let us now verify that the preceding model becomes geometric Brownian motion as we let Δ become smaller and smaller. To begin, let Y_i equal 1 if the price goes up at time $i\Delta$, and let it be 0 if it goes down. Now, the number of times that the security's price goes up in the first n time increments is $\sum_{i=1}^{n} Y_i$, and the number of times it goes down is $n - \sum_{i=1}^{n} Y_i$. Hence, $S(n\Delta)$, its price at the end of this time, can be expressed as

$$S(n\Delta) = S(0)u^{\sum_{i=1}^{n} y_i} d^{n-\sum_{i=1}^{n} y_i}$$

or $S(n\Delta) = d^n S(0)\left(\frac{u}{d}\right)^{\sum_{i=1}^{n} Y_i}$

If we now let $n = t/\Delta$, then the preceding equation can be expressed as

$$\frac{S(t)}{S(0)} = d^{t/\Delta}\left(\frac{u}{d}\right)^{\sum_{i=1}^{t/\Delta} y_i}$$

Taking logarithms gives

$$\log\left(\frac{S(t)}{S(0)}\right) = \frac{t}{\Delta}\log(d) + \log\left(\frac{u}{d}\right)\sum_{i=1}^{t/\Delta} Y_i$$

$$= \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\sum_{i=1}^{\frac{t}{\Delta}} Y_i \quad \ldots\ldots\ldots\ldots\ldots \text{(2.1)}$$

where Equation (2.1) used the definitions of u and d. Now, as Δ goes to 0 , there are more and more terms in the summation $\sum_{i=1}^{t/\Delta} Y_i$; hence, by the central limit theorem, this sum becomes more and more normal, implying from Equation (2.1) that $\log(S(t)/S(0))$ becomes a normal random variable. Moreover, from Equation (2.1) we obtain that

$$\begin{aligned} E\left[\log\left(\frac{S(t)}{S(0)}\right)\right] &= \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\sum_{i=1}^{t/\Delta} E[Y_i] \\ &= \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\frac{t}{\Delta}p \\ &= \frac{-t\sigma}{\sqrt{\Delta}} + \frac{t\sigma}{\sqrt{\Delta}}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right) \\ &= \mu t \end{aligned}$$

Furthermore, Equation (3.1) yields that

$$\begin{aligned} \operatorname{Var}\left(\log\left(\frac{S(t)}{S(0)}\right)\right) &= 4\sigma^2\Delta\sum_{i=1}^{t/\Delta} \operatorname{Var}(Y_i) \quad \text{(by independence)} \\ &= 4\sigma^2 tp(1-p) \\ &\approx \sigma^2 t \ (\text{ since, for small } \Delta, p \approx 1/2) \end{aligned}$$

Thus we see that, as Δt becomes smaller and smaller, $\log(S(t)/S(0))$ (and, by the same reasoning, $\log(S(t+y)/S(y))$) becomes a normal random variable with mean μt and variance $t\sigma^2$.

In addition, because successive price changes are independent and each has the same probability of being an increase, it follows that $S(t+y)/S(y)$ is independent of earlier price changes before time y. Hence, as Δ goes to 0 , both conditions of geometric Brownian motion are met, showing that the model indeed becomes geometric Brownian motion.

2.3 Brownian Motion:

Geometric Brownian motion can be considered to be a variant of a long studied model known as Brownian motion. It is defined as follows.

Definition The collection of prices $S(y), 0 \leq y < \infty$, is said to follow a Brownian motion with drift parameter μ and variance parameter σ^2 if, for all nonegative values of y and t, the random variable $S(t + y) - S(y)$

is independent of all prices up to time y and, in addition, is a normal random variable with mean μt and variance $t\sigma^2$.

Thus, Brownian motion shares with geometric Brownian motion the property that a future price depends on the present and all past prices only through the present price; however, in Brownian motion it is the difference in prices (and not the logarithm of their ratio) that has a normal distribution.

The Brownian motion process has a distinguished scientific pedigree. It is named after the English botanist Robert Brown, who first described (in 1827) the unusual motion exhibited by a small particle that is totally immersed in a liquid or gas.

The first explanation of this motion was given by Albert Einstein in 1905. He showed mathematically that Brownian motion could be explained by assuming that the immersed particle was continually being subjected to bombardment by the molecules of the surrounding medium.

A mathematically concise definition, as well as an elucidation of some of the mathematical properties of Brownian motion, was given by the American applied mathematician Norbert Wiener in a series of papers originating in 1918.

Interestingly, Brownian motion was independently introduced in 1900 by the French mathematician Bachelier, who used it in his doctoral dissertation to model the price movements of stocks and commodities.

However, Brownian motion appears to have two major flaws when used to model stock or commodity prices. First, since the price of a stock is a normal random variable, it can theoretically become negative.

Second, the assumption that a price difference over an interval of fixed length has the same normal distribution no matter what the price at the beginning of the interval does not seem totally reasonable.

For instance, many people might not think that the probability a stock presently selling at \$20 would drop to \$15 (a loss of 25%) in one month would be the same as the probability that when the stock is at \$10 it would drop to \$5 (a loss of 50%) in one month. The geometric Brownian motion model, on the other hand, possesses neither of these flaws. Since it is now the logarithm of the stock's price that is a normal random variable, the model does not allow for negative stock prices.

In addition, since it is the ratios of prices separated by a fixed length of time that have the same distribution, geometric Brownian motion makes what many feel is the more reasonable assumption that it is the percentage change in price, and not the absolute change, whose probabilities do not depend on the present price.

However, it should be noted that - in both of these models - once the model parameters μ and σ are determined, the only information that is needed for predicting future prices is the present price; information about past prices is irrelevant.

Exercises 1:

1. Suppose that $S(y), y \geq 0$, is a geometric Brownian motion with drift parameter $\mu = .01$ and volatility parameter $\sigma = .2$. If $S(0) = 100$, find;

   (a) $E[S(10)]$;

   (b) $P\{S(10) > 100\}$;

   (c) $P\{S(10) < 110\}$.

2. It can be shown that if X is a normal random variable with mean m and variance v^2, then

$$E[e^X] = e^{m+v^2/2}$$

3. Use the result of the preceding exercise to find $\text{Var}(S(t))$ when $S(0) = s_0$.

UNIT-III:

Interest Rate and Present Value Analysis

Chapter 3: Sections: 3.1-3.4

3.Interest Rates and Present Value Analysis

3.1. Interest Rates:

If you borrow the amount P (called the principal), which must be repaid after a time T along with simple interest at rate r per time T, then the amount to be repaid at time T is

$$P + rP = P(1 + r).$$

That is, you must repay both the principal P and the interest, equal to the principal times the interest rate. For instance, if you borrow $\$100$ to be repaid after one year with a simple interest rate of 5% per year (i.e., $r = .05$), then you will have to repay $\$105$ at the end of the year.

Example 3.1.1:

Suppose that you borrow the amount P, to be repaid after one year along with interest at a rate r per year compounded semiannually. What does this mean? How much is owed in a year?

Solution:

In order to solve this example, you must realize that having your interest compounded semiannually means that after half a year you are to be charged simple interest at the rate of $r/2$ per half-year, and that interest is then added on to your principal, which is again charged interest at rate $r/2$ for the second half-year period. In other words, after six months you owe $P(1 + r/2)$

This is then regarded as the new principal for another six-month loan at interest rate $r/2$; hence, at the end of the year you will owe

$$P(1 + r/2)(1 + r/2) = P(1 + r/2)^2.$$

Example 3.1.2:

If you borrow $1,000 for one year at an interest rate of 8% per year compounded quarterly, how much do you owe at the end of the year?

Solution:

An interest rate of 8% that is compounded quarterly is equivalent to paying simple interest at 2% per quarter-year, with each successive quarter charging interest not only on the original principal but also on the interest that has accrued up to that point. Thus, after one quarter you owe

$$1{,}000(1+.02)$$

after two quarters you owe

$$1{,}000(1+.02)(1+.02) = 1{,}000(1+.02)^2$$

after three quarters you owe

$$1{,}000(1+.02)^2(1+.02) = 1{,}000(1+.02)^3$$

and after four quarters you owe

$$1{,}000(1+.02)^3(1+.02) = 1{,}000(1+.02)^4 = \$1{,}082.40$$

Example 3.1.3:

Many credit-card companies charge interest at a yearly rate of 18% compounded monthly. If the amount P is charged at the beginning of a year, how much is owed at the end of the year if no previous payments have been made?

Solution:

Such a compounding is equivalent to paying simple interest every month at a rate of $18/12 = 1.5\%$ per month, with the accrued interest then added to the principal owed during the next month. Hence, after one year you will owe

$$P(1 + .015)^{12} = 1.1956P$$

If the interest rate r is compounded then, as we have seen in Examples **3.1.2** and **3.1.3** , the amount of interest actually paid is greater than if we were paying simple interest at rate r. The reason, of course, is that in compounding we are being charged interest on the interest that has already been computed in previous compounding's. In these cases, we call r the nominal interest rate, and we define the effective interest rate, call it r_{eff} , by

$$r_{\text{eff}} = \frac{\text{amount repaid at the end of a year } - P}{P}$$

For instance, if the loan is for one year at a nominal interest rate r that is to be compounded quarterly, then the effective interest rate for the year is

$$r_{\text{eff}} = (1 + r/4)^4 - 1.$$

Thus, in Example **3.1.2.** the effective interest rate is 8.24% whereas in Example **3.1.3** it is 19.56%. Since

$$P(1 + r_{\text{eff}}) = \text{ amount repaid at the end of a year,}$$

the payment made in a one-year loan with compound interest is the same as if the loan called for simple interest at rate r_{eff} per year.

Suppose now that we borrow the principal P for one year at a nominal interest rate of r per year, compounded continuously. Now, how much is owed at the end of the year? Of course, to answer this we must first decide on an appropriate definition of "continuous" compounding.

To do so, note that if the loan is compounded at n equal intervals in the year, then the amount owed at the end of the year is $P(1 + r/n)^n$.

As it is reasonable to suppose that continuous compounding refers to the limit of this process as n grows larger and larger, the amount owed at time 1 is $P\lim_{n\to\infty}(1+r/n)^n = Pe^r$.

Example 3.1.4:

If a bank offers interest at a nominal rate of 5% compounded continuously, what is the effective interest rate per year?

Solution:

The effective interest rate is

$$r_{\text{cff}} = \frac{Pe^{.05} - P}{P} = e^{.05} - 1 \approx .05127.$$

That is, the effective interest rate is 5.127% per year.
If the amount P is borrowed for t years at a nominal interest rate of r per year compounded continuously, then the amount owed at time t is $\text{P}e^{rt}$. This is seen by interpreting the interest rate as being a continuous compounding of a nominal rate of rt per time t; hence, the amount owed at time t is

$$P\lim_{n\to\infty}(1+rt/n)^n = Pe^{rt}.$$

Example 3.1.5:

The Doubling Rule If you put funds into an account that pays interest at rate r compounded annually, how many years does it take for your funds to double?

Solution:

Since your initial deposit of D will be worth $D(1+r)^n$ after n years, we need to find the value of n such that

$$(1+r)^n = 2$$

Now,

$$(1+r)^n = \left(1+\frac{nr}{n}\right)^n$$
$$\approx e^{nr}$$

where the approximation is fairly precise provided that n is not too small. Therefore,

$$e^{nr} \approx 2$$

implying that

$$n \approx \frac{\log(2)}{r} = \frac{0.693}{r}$$

Thus, it will take n years for your funds to double when

$$n \approx \frac{0.7}{r}$$

For instance, if the interest rate is $1\% (r = 0.01)$ then it will take approximately 70 years for your funds to double; if $r = 0.02$, it will take about 35 years; if $r = 0.03$, it will take about $23\frac{1}{3}$ years; if $r = 0.05$, it will take about 14 years; if $r = 0.07$, it will take about 10 years; and if $r = 0.10$, it will take about 7 years.

As a check on the preceding approximations, note that (to three-decimal-place accuracy):

$$(1.01)^{70} = 2.007$$
$$(1.02)^{35} = 2.000$$
$$(1.03)^{23.33} = 1.993$$
$$(1.05)^{14} = 1.980$$
$$(1.07)^{10} = 1.967$$
$$(1.10)^{7} = 1.949$$

3.2. Present Value Analysis:

Suppose that one can both borrow and loan money at a nominal rate r that is ompounded periodically. Under these conditions, what is the present worth of a payment of v dollars that will be made at the end of period i ? Since a bank loan of $v(1+r)^{-i}$ would require a payoff of v at period i, it follows that the present value of a payoff of v to be made at time period i is $v(1+r)^{-i}$. The concept of present value enables us to compare different income streams to see which is preferable.

Example 3.2.1:

Suppose that you are to receive payments (in thousands of dollars) at the end of each of the next five years. Which of the following three payment sequences is preferable?

A. 12,14,16,18,20;

B. 16,16,15,15,15;

C. 20,16,14,12,10.

Solution:

If the nominal interest rate is r compounded yearly, then the present value of the sequence of payments

$x_i(i=1,2,3,4,5)$ is $\sum_{i=1}^{5} (1+r)^{-i}x_i$

the sequence having the largest present value is preferred.

It thus follows that the superior sequence of payments depends on the interest rate. If r is small, then the sequence **A** is best since its sum of payments is the highest.

For a somewhat larger value of r, the sequence **B** would be best because - although the total of its payments (77) is less than that of **A**(80) - its earlier payments are larger than are those of **A**.

For an even larger value of r, the sequence **C**, whose earlier payments are higher than those of either **A** or **B**, would be best. Table 3.1 gives the present values of these payment streams for three different values of r.

It should be noted that the payment sequences can be compared according to their values at any specified time. For instance, to compare them in terms of their time- 5 values, we would determine which sequence of payments yields the largest value of

Table 3.1: Present Values

	Payment Sequence		
r	**A**	**B**	**C**
.1	59.21	58.60	56.33
.2	45.70	46.39	45.69
.3	36.49	37.89	38.12

$$\sum_{i=1}^{5}(1+r)^{5-i}x_i = (1+r)^5\sum_{i=1}^{5}(1+r)^{-i}x_i$$

Consequently, we obtain the same preference ordering as a function of interest rate as before.

Remark:

Let $\mathbf{a} = (a_0, a_1, \dots, a_n)$ and $\mathbf{b} = (b_0, b_1, \dots, b_n)$ be cash flow sequences, and suppose that the present value of the a sequence is at least as large as that of the **b** sequence when the interest rate is r. That is,

$$\text{PV}(\mathbf{a}) = \sum_{i=0}^{n} a_i(1+r)^{-i} \geq \sum_{i=0}^{n} b_i(1+r)^{-i} = \text{PV}(\mathbf{b})$$

One way of seeing the superiority of the a sequence is to note that it can be transformed, by borrowing and saving at the rate r, into a cash flow sequence $\mathbf{c} = (c_0, c_1, \ldots, c_n)$ having $c_i \geq b_i$ for each $i = 0, \ldots, n$.

We prove this fact by induction on n. As it is immediate when $n = 0$, assume that the result holds whenever the cash flow sequences are of length n, and now consider cash flow sequences **a** and **b** that are of length $n + 1$ and are such that the present value of a is greater than or equal to that of **b**. There are two cases to consider.

Case 1:

$a_0 \geq b_0$. In this case, start by putting aside the amount b_0 and depositing $a_0 - b_0$ in a bank to be withdrawn in the next period. In this manner, **a** is transformed into the cash flow sequence

$(b_0, (1 + r)(a_0 - b_0) + a_1, \ldots, a_n)$

Now, since

$$(1 + r)(a_0 - b_0) + \sum_{i=0}^{n-1} a_{i+1}(1 + r)^{-i} - \sum_{i=0}^{n-1} b_{i+1}(1 + r)^{-i}$$
$$= (1 + r)[\mathrm{PV}(\mathbf{a}) - \mathrm{PV}(\mathbf{b})] \geq 0,$$

it follows that the time-1 value of the cash flows $(1 + r)(a_0 - b_0) + a_1, \ldots, a_n$ to be received in periods $1, \ldots, n$ is at least as large as that of the cash flows $b_1, \ldots, b_n$. Hence, by the induction hypothesis we can transform the cash flow sequence

$(b_0, (1 + r)(a_0 - b_0) + a_1, \ldots, a_n)$ into a sequence ($b_0, c_1, \ldots, c_n$) which is such that $c_i \geq b_i$ for each i. This completes the induction proof in this case.

Case 2:

$a_0 < b_0$. In this case, start by borrowing $b_0 - a_0$, to be repaid in period 1. This transforms the cash flow sequence a into the sequence $(b_0, a_1 - (1 + r)(b_0 - a_0), a_2, \ldots, a_n)$. It easily follows that the time-1 value of the cash flows $a_1 - (1 + r)(b_0 - a_0), a_2, \ldots, a_n$ to be received at the ends of periods $1, \ldots, n$ is at least as great as that of the cash flows $b_1, \ldots, b_n$, so the result again follows from the induction hypothesis.

Example 3.2.2:

A company needs a certain type of machine for the next five years. They presently own such a machine, which is now worth $\$6{,}000$ but will lose $\$2{,}000$ in value in each of the next three years, after which it will be worthless and unuseable. The (beginning-of-the-year) value of its yearly operating cost is $\$9{,}000$, with this amount expected to increase by $\$2{,}000$ in each subsequent year that it is used. A new machine can be purchased at the beginning of any year for a fixed cost of $\$22{,}000$. The lifetime of a new machine is six years, and its value decreases by $\$3{,}000$ in each of its first two years of use and then by $\$4{,}000$ in each following year. The operating cost of a new machine is $\$6{,}000$ in its first year, with an increase of $\$1{,}000$ in each subsequent year. If the interest rate is 10%, when should the company purchase a new machine?

Solution:

The company can purchase a new machine at the beginning of year 1,2,3, or 4, with the following six-year cash flows (in units of $\$1{,}000$) as a result:

- buy at beginning of year $1 - 22,7,8,9,10,-4$;
- buy at beginning of year $2 - 9,24,7,8,9,-8$;
- buy at beginning of year $3 - 9,11,26,7,8,-12$;
- buy at beginning of year $4 - 9,11,13,28,7,-16$.

To see why this listing is correct, suppose that the company will buy a new machine at the beginning of year 3. Then its year-1 cost is the $\$9{,}000$ operating cost of the old machine; its year-2 cost is the $\$11{,}000$ operating cost of this machine; its year -3 cost is the $\$22{,}000$ cost of a new machine, plus the $\$6{,}000$ operating cost of this machine, minus the $\$2{,}000$ obtained for the replaced machine; its year-4 cost is the $\$7{,}000$ operating cost; its year- 5 cost is the $\$8{,}000$

operating cost; and its year- 6 cost is $-\$12{,}000$, the negative of the value of the 3 -year-old machine that it no longer needs. The other cash flow sequences are similarly argued.

With the yearly interest rate $r = 0.10$, the present value of the first cost-flow sequence is

$$22 + \frac{7}{1.1} + \frac{8}{(1.1)^2} + \frac{9}{(1.1)^3} + \frac{10}{(1.1)^4} - \frac{4}{(1.1)^5} = 46.083$$

The present values of the other cash flows are similarly determined, and the four present values are

46.083,43.794,43.760,45.627

Therefore, the company should purchase a new machine two years from now.

Example 3.2.3:

An individual who plans to retire in 20 years has decided to put an amount A in the bank at the beginning of each of the next 240 months, after which she will withdraw $\$1{,}000$ at the beginning of each of the following 360 months. Assuming a nominal yearly interest rate of 6% compounded monthly, how large does A need to be?

Solution:

Let $r = .06/12 = .005$ be the monthly interest rate. With $\beta = \frac{1}{1+r}$, the present value of all her deposits is

$$A + A\beta + A\beta^2 + \cdots + A\beta^{239} = A\frac{1-\beta^{240}}{1-\beta}$$

Similarly, if W is the amount withdrawn in the following 360 months, then the present value of all these withdrawals is

$$W\beta^{240} + W\beta^{241} + \cdots + W\beta^{599} = W\beta^{240}\frac{1-\beta^{360}}{1-\beta}$$

Thus she will be able to fund all withdrawals (and have no money left in her account) if

$$A\frac{1-\beta^{240}}{1-\beta} = W\beta^{240}\frac{1-\beta^{360}}{1-\beta}$$

With $W = 1{,}000$, and $\beta = 1/1.005$, this gives

$$A = 360.99$$

That is, saving \$361 a month for 240 months will enable her to withdraw $\$1{,}000$ a month for the succeeding 360 months.

Remark:

In this example we have made use of the algebraic identity

$$1 + b + b^2 + \cdots + b^n = \frac{1-b^{n+1}}{1-b}$$

We can prove this identity by letting

$$x = 1 + b + b^2 + \cdots + b^n$$

and then noting that

$$\begin{aligned} x - 1 &= b + b^2 + \cdots + b^n \\ &= b(1 + b + \cdots + b^{n-1}) \\ &= b(x - b^n) \end{aligned}$$

Therefore,

$$(1-b)x = 1 - b^{n+1}$$

which yields the identity.

Example 3.2.4:

Suppose you have just spoken to a bank about borrowing $\$100{,}000$ to purchase a house, and the loan officer has told you that a $\$100{,}000$ loan, to be repaid in monthly installments over 15 years with an interest rate of $.6\%$ per month, could be arranged. If the bank charges a loan initiation fee of $\$600$, a house inspection fee of $\$400$, and 1 "point," what is the effective annual interest rate of the loan being offered?

Solution:

To begin, let us determine the monthly mortgage payment, call it A, of such a loan. Since $\$100{,}000$ is to be repaid in 180 monthly payments at an interest rate of $.6\%$ per month, it follows that

$$A[\alpha + \alpha^2 + \cdots + \alpha^{180}] = 100{,}000$$

where $\alpha = 1/1.006$. Therefore,

$$A = \frac{100{,}000(1-\alpha)}{\alpha(1-\alpha^{180})} = 910.05$$

So if you were actually receiving $\$100{,}000$ to be repaid in 180 monthly payments of $\$910.05$, then the effective monthly interest rate would be $.6\%$.

However, taking into account the initiation and inspection fees involved and the bank charge of 1 point (which means that 1% of the nominal loan of $\$100{,}000$ must be paid to the bank when the loan is received), it follows that you are actually receiving only $\$98{,}000$.

Consequently, the effective monthly interest rate is that value of r such that

$$A[\beta + \beta^2 + \cdots + \beta^{180}] = 98{,}000$$

where $\beta = (1+r)^{-1}$. Therefore,

$$\frac{\beta(1-\beta^{180})}{1-\beta} = 107.69$$

or, since $\frac{1-\beta}{\beta} = r$,

$$\frac{1-\left(\frac{1}{1+r}\right)^{180}}{r} = 107.69$$

Numerically solving this by trial and error (easily accomplished since we know that $r > .006$) yields the solution

$$r = .00627$$

Since $(1+.00627)^{12} = 1.0779$, it follows that what was quoted as a monthly interest rate of $.6\%$ is, in reality, an effective annual interest rate of approximately 7.8%.

Example 3.2.5:

Suppose that one takes a mortgage loan for the amount L that is to be paid back over n months with equal payments of A at the 48 Interest Rates and Present Value Analysis
end of each month. The interest rate for the loan is r per month, compounded monthly.
(a) In terms of L, n, and r, what is the value of A ?
(b) After payment has been made at the end of month j, how much additional loan principal remains?
(c) How much of the payment during month j is for interest and how much is for principal reduction? (This is important because some contracts allow for the loan to be paid back early and because the interest part of the payment is tax-deductible.)

Solution:

The present value of the n monthly payments is

$$\frac{A}{1+r}+\frac{A}{(1+r)^2}+\cdots+\frac{A}{(1+r)^n}=\frac{A}{1+r}\frac{1-\left(\frac{1}{1+r}\right)^n}{1-\frac{1}{1+r}}$$
$$=\frac{A}{r}[1-(1+r)^{-n}].$$

Since this must equal the loan amount L, we see that

$$A=\frac{Lr}{1-(1+r)^{-n}}=\frac{L(\alpha-1)\alpha^n}{\alpha^n-1} \quad \ldots\ldots\ldots\ldots\ldots.. (3.1)$$

Where $\alpha=1+r$.

For instance, if the loan is for $\$100{,}000$ to be paid back over 360 months at a nominal yearly interest rate of .09 compounded monthly, then $r=.09/12=.0075$ and the monthly payment (in dollars) would be

$$A=\frac{100{,}000(.0075)(1.0075)^{360}}{(1.0075)^{360}-1}=804.62$$

Let R_j denote the remaining amount of principal owed after the payment at the end of month $j(j=0,\ldots,n)$. To determine these quantities, note that if one owes R_j at the end of month j then the amount owed immediately before the payment at the end of month $j+1$ is $(1+r)R_j$; because one then pays the amount A, it follows that

$$R_{j+1}=(1+r)R_j-A=\alpha R_j-A$$

Starting with $R_0=L$, we obtain:

$$\begin{aligned}
R_1&=\alpha L-A\\
R_2&=\alpha R_1-A\\
&=\alpha(\alpha L-A)-A\\
&=\alpha^2L-(1+\alpha)A,\\
R_3&=\alpha R_2-A\\
&=\alpha(\alpha^2L-(1+\alpha)A)-A\\
&=\alpha^3L-(1+\alpha+\alpha^2)A.
\end{aligned}$$

In general, for $j = 0, \ldots, n$ we obtain

$$\begin{aligned} R_j &= \alpha^j L - A\left(1 + \alpha + \cdots + \alpha^{j-1}\right) \\ &= \alpha^j L - A\frac{\alpha^j - 1}{\alpha - 1} \\ &= \alpha^j L - \frac{L\alpha^n\left(\alpha^j - 1\right)}{\alpha^n - 1} \text{ (from equation(3.1))} \\ &= \frac{L\left(\alpha^n - \alpha^j\right)}{\alpha^n - 1}. \end{aligned}$$

Let I_j and P_j denote the amounts of the payment at the end of month j that are for interest and for principal reduction, respectively. Then, since R_{j-1} was owed at the end of the previous month, we have

$$\begin{aligned} I_j &= rR_{j-1} \\ &= \frac{L(\alpha - 1)\left(\alpha^n - \alpha^{j-1}\right)}{\alpha^n - 1} \end{aligned}$$

and

$$\begin{aligned} P_j &= A - I_j \\ &= \frac{L(\alpha - 1)}{\alpha^n - 1}\left[\alpha^n - \left(\alpha^n - \alpha^{j-1}\right)\right] \\ &= \frac{L(\alpha - 1)\alpha^{j-1}}{\alpha^n - 1}. \end{aligned}$$

As a check, note that

Interest Rates and Present Value Analysis

$$\sum_{j=1}^{n} P_j = L$$

It follows that the amount of principal repaid in succeeding months increases by the factor $\alpha = 1 + r$. For example, in a $\$100{,}000$ loan for 30 years at a nominal interest rate of 9% per year compounded monthly, only \$54.62 of the \$804.62 paid during the first month goes toward reducing the principal of the loan; the remainder is interest. In each succeeding month, the amount of the payment that goes toward the principal increases by the factor 1.0075.

Consider two cash flow sequences,

$b_1, b_2, \ldots, b_n$ and $c_1, c_2, \ldots, c_n$

Under what conditions is the present value of the first sequence at least as large as that of the second for every positive interest rate r ? Clearly, $b_i \geq c_i (i = 1, \ldots, n)$ is a sufficient condition. However, we can obtain weaker sufficient conditions. Let

$$B_i = \sum_{j=1}^{i} b_j \text{ and } C_i = \sum_{j=1}^{i} c_j \text{ for } i = 1, \ldots, n$$

then it can be shown that the condition

$B_i \geq C_i$ for each $i = 1, \ldots, n$

suffices. An even weaker sufficient condition is given by the following proposition.

Proposition 3.2.1:

If $B_n \geq C_n$ and if

$$\sum_{i=1}^{k} B_i \geq \sum_{i=1}^{k} C_i$$

for each $k = 1, \ldots, n$, then

$$\sum_{i=1}^{n} b_i (1+r)^{-i} \geq \sum_{i=1}^{n} c_i (1+r)^{-i}$$

for every $r > 0$.

In other words, Proposition 2.5.1. states that the cash flow sequence $b_1, \ldots, b_n$ will, for every positive interest rate r, have a larger present value than the cash flow sequence $c_1, \ldots, c_n$ if (i) the total of the b cashflows is at least as large as the total of the c-cashflows and

(ii) for every $k = 1, \ldots, n$,

$$kb_1 + (k-1)b_2 + \cdots + b_k \geq kc_1 + (k-1)c_2 + \cdots + c_k$$

3.3. Rate of Return:

Consider an investment that, for an initial payment of $a (a > 0)$, returns the amount b after one period. The rate of return on this investment is defined to be the interest rate r that makes the present value of the return equal to the initial payment.

That is, the rate of return is that value r such that

$$\frac{b}{1+r} = a \text{ or } r = \frac{b}{a} - 1$$

Thus, for example, a $\$100$ investment that returns $\$150$ after one year is said to have a yearly rate of return of .50 .

More generally, consider an investment that, for an initial payment of $a (a > 0)$, yields a string of nonnegative returns $b_1, \ldots, b_n$. Here b_i is to be received at the end of period $i (i = 1, \ldots, n)$, and $b_n > 0$. We define the rate of return per period of this investment to be the value of the interest rate such that the present value of the cash flow sequence is equal to zero when values are compounded periodically at that interest rate. That is, if we define the function P by

$$P(r) = -a + \sum_{i=1}^{n} b_i (1+r)^{-i} \quad \ldots\ldots\ldots\ldots\ldots (3.2)$$

then the rate of return per period of the investment is that value $r^* > -1$ for which

$$P(r^*) = 0.$$

It follows from the assumptions $a > 0, b_i \geq 0$, and $b_n > 0$ that $P(r)$ is a strictly decreasing function of r when $r > -1$, implying (since $\lim_{r \to -1} P(r) = \infty$ and $\lim_{r \to \infty} P(r) = -a < 0$) that there is a unique value r^* satisfying the preceding equation. Moreover, since

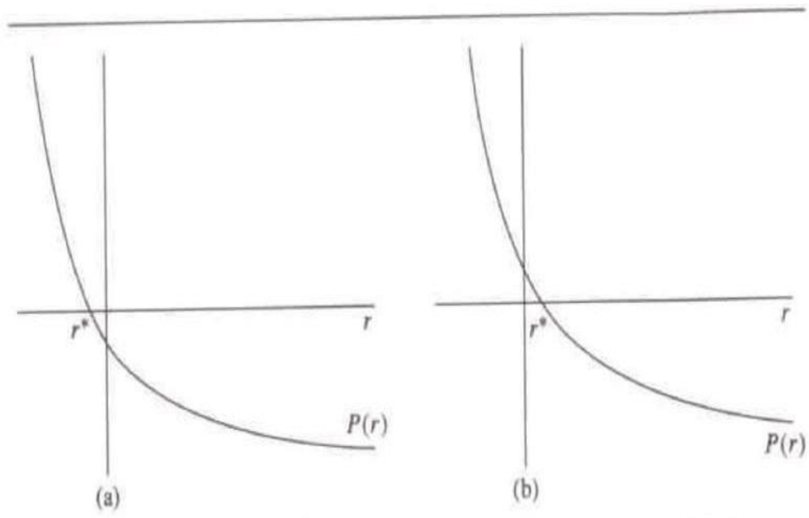

Figure 3.1: $P(r) = -a + \sum_{i \geq 1} b_i (1 + r)^{-i}$: (a) $\sum_i b_i < a$; (b) $\sum_i b_i > a$

$$P(0) = \sum_{i=1}^{n} b_i - a$$

it follows (see Figure 3.1) that r^* will be positive if

$$\sum_{i=1}^{n} b_i > a$$

and that r^* will be negative if

$$\sum_{i=1}^{n} b_i < a$$

That is, there is a positive rate of return if the total of the amounts received exceeds the initial investment, and there is a negative rate of return if the reverse holds.

Moreover, because of the monotonicity of $P(r)$, it follows that the cash flow sequence will have a positive present value when the interest rate is less than r^* and a negative present value when the interest rate is greater than r^*.

When an investment's rate of return is r^* per period, we often say that the investment yields a $100r^*$-percent rate of return per period.

Example 3.3.1:

Find the rate of return from an investment that, for an initial payment of 100, yields return of 60 at the end of each of the first two periods.

Solution:

The rate of return will be the solution to

$$100 = \frac{60}{1+r} + \frac{60}{(1+r)^2}$$

Letting $x = 1/(1+r)$, the preceding can be written as

$$60x^2 + 60x - 100 = 0$$

which yields that

$$x = \frac{-60 \pm \sqrt{60^2 + 4(60)(100)}}{120}$$

Since $-1 < r$ implies that $x > 0$, we obtain the solution

$$x = \frac{\sqrt{27{,}600} - 60}{120} \approx 0.8844$$

Hence, the rate of return r^* is such that

$$1 + r^* \approx \frac{1}{.8844} \approx 1.131$$

That is, the investment yields a rate of return of approximately 13.1% per period.

The rate of return of investments whose string of payments spans more than two periods will usually have to be numerically determined. Because of the monotonicity of $P(r)$, a trial-and-error approach is usually quite efficient.

Remarks:

(1) If we interpret the cash flow sequence by supposing that $b_1, \dots, b_n$ represent the successive periodic payments made to a lender who loans a_0 to a borrower, then the lender's periodic rate of return r^* is exactly the effective interest rate per period paid by the borrower. (2) The quantity r^* is also sometimes called the internal rate of return. Consider now a more general investment cash flow sequence $c_0, c_1, \dots, c_n$. Here, if $c_i \geq 0$ then the amount c_i is received by the investor at the end of period i, and if $c_i < 0$ then the amount $-c_i$ must be paid by the investor at the end of period i. If we let $P(r) = \sum_{i=0}^{n} c_i(1+r)^{-i}$

be the present value of this cash flow when the interest rate is r per period, then in general there will not necessarily be a unique solution of the equation $P(r) = 0$ in the region $r > -1$. As a result, the rate-of-return concept is unclear in the case of more general cash flows than the ones considered here. In addition, even in cases where we can show that the preceding equation has a unique solution r^*, it may result that $P(r)$ is not a monotone function of r; consequently, we could not assert that the investment yields a positive present value return when the interest rate is on one side of r^* and a negative present value return when it is on the other side.

One general situation for which we can prove that there is a unique solution is when the cash flow sequence starts out negative (resp. positive), eventually becomes positive (negative), and then remains nonnegative (non-positive) from that point on. In other words, the sequence $c_0, c_1, \dots, c_n$

has a single sign change. It then follows - upon using Descartes' rule of sign, along with the known existence of at least one solution - that there is a unique solution of the equation $P(r) = 0$ in the region $r > -1$.

3.4. Continuously Varying Interest Rates:

Suppose that interest is continuously compounded but with a rate that is changing in time. Let the present time be time 0 , and let $r(s)$ denote the interest rate at time s. Thus, if you put x in a bank at time s, then the amount in your account at time $s + h \approx x(1 + r(s)h)$ (h small).

The quantity $r(s)$ is called the spot or the instantaneous interest rate at time s.

Let $D(t)$ be the amount that you will have on account at time t if you deposit 1 at time 0 . In order to determine $D(t)$ in terms of the interest rates $r(s), 0 \leq s \leq t$, note that (for h small)

we have $D(s + h) \approx D(s)(1 + r(s)h)$

$$\Rightarrow D(s + h) - D(s) \approx D(s)r(s)h$$

$$\Rightarrow \frac{D(s + h) - D(s)}{h} \approx D(s)r(s)$$

The preceding approximation becomes exact as h becomes smaller and smaller. Hence, taking the limit as $h \to 0$, it follows that

$$D'(s) = D(s)r(s)$$

$$\Rightarrow \frac{D'(s)}{D(s)} = r(s)$$

implying that

$$\int_0^t \frac{D'(s)}{D(s)} ds = \int_0^t r(s) ds$$

$$\Rightarrow \log(D(t)) - \log(D(0)) = \int_0^t r(s)ds$$

Since $D(0) = 1$, we obtain from the preceding equation that

$$D(t) = \exp\left\{\int_0^t r(s)ds\right\}$$

Now let $P(t)$ denote the present (i.e. time- 0) value of the amount 1 that is to be received at time $t(P(t)$ would be the cost of a bond that yields a return of 1 at time t; it would equal e^{-rt} if the interest rate were always equal to r). Because a deposit of $1/D(t)$ at time 0 will be worth 1 at time t, we see that

$$P(t) = \frac{1}{D(t)} = \exp\left\{-\int_0^t r(s)ds\right\} \qquad \text{………… (3.3)}$$

Let $\bar{r}(t)$ denote the average of the spot interest rates up to time t; that is,

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)ds$$

The function $\bar{r}(t), t \geq 0$, is called the yield curve.

Example 3.4.1:

Find the yield curve and the present value function if $r(s) = \frac{1}{1+s}r_1 + \frac{s}{1+s}r_2$

Solution:

Rewriting $r(s)$ as $r(s) = r_2 + \frac{r_1 - r_2}{1+s}, s \geq 0$

shows that the yield curve is given by

$$\bar{r}(t) = \frac{1}{t}\int_0^t \left(r_2 + \frac{r_1 - r_2}{1+s}\right) ds$$
$$= r_2 + \frac{r_1 - r_2}{t}\log(1+t)$$

Consequently, the present value function is

$$P(t) = \exp\{-t\bar{r}(t)\}$$
$$= \exp\{-r_2 t\}\exp\{-\log((1+t)^{r_1 - r_2})\}$$
$$= \exp\{-r_2 t\}(1+t)^{r_2 - r_1}$$

Exercises 1:

1.What is the effective interest rate when the nominal interest rate of 10% is

(a) compounded semiannually;

(b) compounded quarterly;

(c) compounded continuously?

2. Suppose that you deposit your money in a bank that pays interest at a nominal rate of 10% per year. How long will it take for your money to double if the interest is compounded continuously?

3. If you receive 5% interest compounded yearly, approximately how many years will it take for your money to quadruple? What if you were earning only 4% ?

4. Give a formula that approximates the number of years it would take for your funds to triple if you received interest at a rate r compounded yearly.

5. How much do you need to invest at the beginning of each of the next 60 months in order to have a value of \$100,000 at the end of 60 months, given that the annual nominal interest rate will be fixed at 6% and will be compounded monthly?

6. The yearly cash flows of an investment are $-1{,}000, -1{,}200, 800, 900, 800$.

Is this a worthwhile investment for someone who can both borrow and save money at the yearly interest rate of 6% ?

7. Consider two possible sequences of end of year returns:

20,20,20,15,10,5 and 10,10,15,20,20,20.

Which sequence is preferable if the interest rate, compounded annually, is: (a) 3%; (b) 5%; (c) 10% ?

8.A five-year $10,000 bond with a 10% coupon rate costs $10,000 and pays its holder $500 every six months for five years, with a final additional payment of $10,000 made at the end of those 10 payments. Find its present value if the interest rate is: (a) 6%; (b) 10%; (c) 12%.

9.A friend purchased a new sound system that was selling for $4,200. He agreed to make a down payment of $1,000 and to make 24 monthly payments of $160, beginning one month from time of purchase. What is the effective interest rate being paid?

10. Suppose you have agreed to a bank loan of $120,000, for which the bank charges no fees but 2 points. The quoted interest rate is .5% per month. You are required to pay only the accumulated interest each month for the next 36 months, at which point you must make a balloon payment of the still-owed $120,000. What is the effective interest rate of this loan?

11.You can pay off a loan either by paying the entire amount of $16,000 now or you can pay $10,000 now and $10,000 at the end of ten years. Which is preferable when the nominal continuously compounded interest rate is: (a) 2%; (b) 5%; (c) 10% ?

12.A U.S. treasury bond (selling at a par value of $1,000) that matures at the end of five years is said to have a coupon rate of 6% if, after paying $1,000, the purchaser receives $30 at the end of each of the following nine six-month periods and then receives $1,030 at the end of the tenth period. That is, the bond pays a simple interest rate of 3% per six-month period, with the principal repaid at the end of five years. Assuming a continuously compounded interest rate of 5%, find the present value of such a stream of cash payments.

13. A zero coupon rate bond having face value F pays the bondholder the amount F when the bond matures. Assuming a continuously compounded interest rate of 8%, find the present value of a zero coupon bond with face value $F = 1{,}000$ that matures at the end of ten years.

14. Find the rate of return of a two-year investment that, for an initial payment of 1,000, gives a return at the end of the first year of 500 and a return at the end of the second year of: (a) 300 ; (b) 500 ; (c) 700 .

15.The inflation rate is defined to be the rate at which prices as a whole are increasing. For instance, if the yearly inflation rate is 4% then what cost \$100 last year costs \$104 this year. Let r_i denote the inflation rate, and consider an investment whose rate of return is r. We are often interested in determining the investment's rate of return from the point of view of how much the investment increases one's purchasing power; we call this quantity the investment's inflation-adjusted rate of return and denote it as r_a. Since the purchasing power of the amount $(1+r)x$ one year from now is equivalent to that of the amount $(1+r)x/(1+r_i)$ today, it follows that - with respect to constant purchasing power units - the investment transforms (in one time period) the amount x into the amount $(1+r)x/(1+r_i)$. Consequently, its inflationadjusted rate of return is

$$r_a = \frac{1+r}{1+r_i} - 1$$

When r and r_i are both small, we have the following approximation: $r_a \approx r - r_i$

For instance, if a bank pays a simple interest rate of 5% when the inflation rate is 3%, the inflation-adjusted interest rate is approximately 2%. What is its exact value?

UNIT-IV:

Pricing Contracts via Arbitrage

Chapter 4: Sections: 4.1-4.2

4. Pricing Contracts via Arbitrage

4.1 An Example in Options Pricing:

Suppose that the nominal interest rate is r, and consider the following model for pricing an option to purchase a stock at a future time at a fixed price. Let the present price (in dollars) of the stock be 100 per share, and suppose that we know that, after one-time period, its price will be either 200 or 50 (see Figure 4.1).

Suppose further that, for any y, at a cost of cy you can purchase at time 0 the option to buy y shares of the stock at time 1 at a price of 150 per share.

Thus, for instance, if you purchase this option and the stock rises to 200 , you would then exercise the option at time 1 and realize a gain of $200 - 150 = 50$ for each of the y options purchased. On the other hand, if the price of the stock at time 1 is 50 then the option would be worthless.

In addition to the options, you may also purchase x shares of the stock at time 0 at a cost of $100x$, and each share would be worth either 200 or 50 at time 1. We will suppose that both x and y can be positive, negative, or zero.

That is, you can either buy or sell both the stock and the option. For instance, if x were negative then you would be selling $-x$ shares of stock, yielding you an initial return of $-100x$, and you would then be responsible for buying and returning $-x$ shares of the stock at time 1 at a (time-1) cost of either 200 or 50 per share.

(When you sell a stock that you do not own, we say that you are selling it short.) We are interested in determining the appropriate value of c, the unit cost of an option. Specifically, we will show that if r is the one period interest rate then, unless

$c=[100-50(1+r)^{-1}]/3$, there is a combination of purchases that will always result in a positive present value gain. To show this, suppose that at time 0 we

(a) purchase x units of stock, and

(b) purchase y units of options,

where x and y (both of which can be either positive or negative) are to be determined. The cost of this transaction is $100x+cy$; if this amount

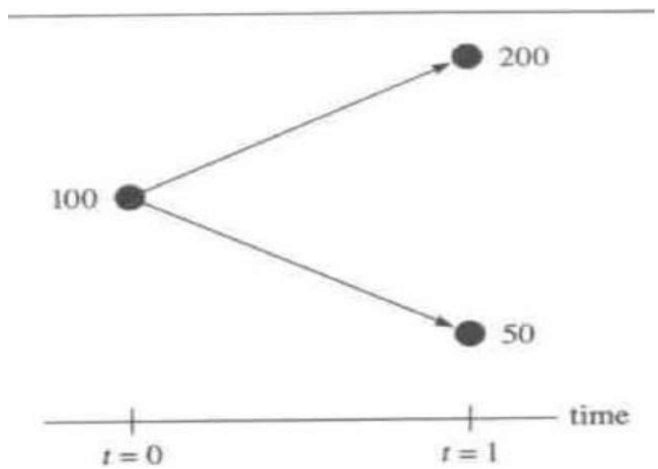

Figure 4.1: Possible Stock Prices at Time 1

is positive, then it should be borrowed from a bank, to be repaid with interest at time 1 ; if it is negative, then the amount received, $-(100x+cy)$, should be put in the bank to be withdrawn at time 1. The value of our holdings at time 1 depends on the price of the stock at that time and is given by

$$\text{value} = \begin{cases} 200x+50y & \text{if the price is } 200 \\ 50x & \text{if the price is } 50 \end{cases}$$

This formula follows by noting that, if the stock's price at time 1 is 200, then the x shares of the stock are worth $200x$ and the y units of options to buy the stock at a share price of 150 are worth $(200-150)y$. On the other hand, if the stock's price is 50 , then the x shares are worth $50x$ and the y units of options are worthless. Now, suppose we choose y so that the value of our holdings at time 1 is the same no matter what the price of the stock at that time. That is, we choose y so that

$$200x + 50y = 50x$$

$$\Rightarrow y = -3x$$

Note that y has the opposite sign of x; thus, if $x > 0$ and so x shares of the stock are purchased at time 0 , then $3x$ units of stock options are also sold at that time. Similarly, if x is negative, then $-x$ shares are sold and $-3x$ units of stock options are purchased at time 0 .

Thus, with $y = -3x$, the

$$\text{time-1 value of holdings} = 50x$$

no matter what the value of the stock. As a result, if $y = -3x$ it follows that, after paying off our loan (if $100x + cy > 0$) or withdrawing our money from the bank (if $100x + cy < 0$), we will have gained the amount

$$\begin{aligned} \text{gain} &= 50x - (100x + cy)(1 + r) \\ &= 50x - (100x - 3xc)(1 + r) \\ &= (1 + r)x[3c - 100 + 50(1 + r)^{-1}] \end{aligned}$$

Thus, if $3c = 100 - 50(1 + r)^{-1}$, then the gain is 0 .

On the other hand, if $3c \neq 100 - 50(1 + r)^{-1}$, then we can guarantee a positive gain (no matter what the price of the stock at time 1) by letting x be positive when $3c > 100 - 50(1 + r)^{-1}$ and by letting x be negative when $3c < 100 - 50(1 + r)^{-1}$.

For instance, if $(1 + r)^{-1} = 0.9$ and the cost per option is $c = 20$, then purchasing one share of the stock and selling three units of options initially costs us $100 - 3(20) = 40$, which is borrowed from the bank. However, the value of this holding at time 1 is 50 whether the stock price rises to 200 or falls to 50. Using $40(1 + r) = 44.44$ of this amount to pay our bank loan results in a guaranteed gain of 5.56. Similarly, if the cost of an option is 15 , then selling one share of the stock

$(x = -1)$ and buying three units of options results in an initial gain of $100 - 45 = 55$, which is put into a bank to be worth $55(1 + r) = 61.11$ at time 1 .

Because the value of our holding at time 1 is -50, a guaranteed profit of 11.11 is attained. A sure-win betting scheme is called an arbitrage.

Thus, for the numbers considered, the only option cost c that does not result in an arbitrage is

$$c = (100 - 45)/3 = 55/3.$$

Remark:

We can also use arbitrage to determine the cost of the option by replicating the option by a combination of borrowing and purchasing the security. To do so, note that if you buy the option at a cost c, then the return from the option at time 1 is

$$\text{return} = \begin{cases} 50 & \text{if the stock's price is } 200 \\ 0 & \text{if the stock's price is } 50 \end{cases}$$

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Now, suppose that at time 0 you borrow from a bank the amount

$$\frac{50}{1+r}y$$

and put up from your own funds the amount

$$\left(100 - \frac{50}{1+r}\right)y$$

to purchase y shares of the stock. At time 1 you will owe the bank $50y$; so, after selling the stock and paying off this debt, you will have a return given by

$$\text{return} = \begin{cases} 150y & \text{if the stock's price is } 200 \\ 0 & \text{if the stock's price is } 50 \end{cases}$$

Therefore, when $y = 1/3$, the return at time 1 is the same as that from the option. In other words, by paying the initial amount $\left(100 - \frac{50}{1+r}\right)/3$ (and borrowing the remaining amount needed to purchase $1/3$ shares of stock) you can replicate the payoff of the option. These two investment returns are identical, so their costs must be also if one is to avoid an arbitrage.

Thus, if $c > \left(100 - \frac{50}{1+r}\right)/3$, then a sure profit can be made by using your own funds along with borrowed money to purchase $1/3$ shares of stock while simultaneously selling one option share. If $c < \left(100 - \frac{50}{1+r}\right)/3$, then a sure profit can be made by selling $1/3$ shares of the stock, using c of the returns of this sale to purchase the option, and depositing the remaining $100/3 - c$ in a bank.

4.2. Other Examples of Pricing via Arbitrage:

The type of option considered in Section 4.1 is known as a call option because it gives one the option of calling for the stock at a specified price, known as the exercise or strike price.

An American style call option allows the buyer to exercise the option at any time up to the expiration time, whereas a European style call option can only be exercised at the expiration time. Although it might seem that, because of its additional flexibility, the American style option should be worth more, it turns out that it is never optimal to exercise a call option early; thus, the two style options have identical worth's. We now prove this claim.

Proposition 4.2.1

One should never exercise an American style call option before its expiration time t.

Proof:

Suppose that the present price of the stock is S, that you own an option to buy one share of the stock at a fixed price K, and that the option expires after an additional time t. As soon as you exercise the option, you will realize the amount $S - K$. However, consider what would transpire if, instead of exercising the option, you sell the stock short and then purchase the stock at the exercise time, either by paying the market price at that time or by exercising your option and paying K, whichever is less expensive. Under this strategy, you will initially receive S, and will then have to pay the minimum of the market price and the exercise price K after an additional time t. This is clearly preferable to receiving S and immediately paying out K.

Our next example uses arbitrage considerations to derive a lower bound on the cost of a call option.

Example 4.2.1:

Let C denote the cost of an option to purchase a security at time t for the strike price K. If S is the present price of the security and r is the continuously compounded interest rate, then

$$C \geq S - Ke^{-rt}$$

To see why this must be true, suppose to the contrary that

$$C < S - Ke^{-rt}$$

or (equivalently) that

$$(S - C)e^{rt} > K$$

However, if the foregoing inequality held then we could guarantee a sure win by

(1) buying the option and

(2) selling the security.

These two transactions will lead to a cash gain of $S - C$, which we can use to purchase a bond that matures at time t. At time t, we then return the security previously sold by buying it for a cost equal to the minimum of the strike price K of the option we own or the market price at time t.

Consequently, at time t we will receive $(S - C)e^{rt}$ from the bond, and then use at most K of this amount to purchase the security, thus yielding us a positive gain of at least

$$(S - C)e^{rt} - K$$

In addition to call options there are also put options on stocks. These give their owners the option of putting a stock up for sale at a specified price. An American style put option allows the owner to put the stock up for sale - that is, to exercise the option - at any time up to the expiration time of the option. A European style put option can only be exercised at its expiration time. Contrary to the situation with call options, it may be advantageous to exercise a put option before its expiration time, and so the American style put option may be worth more than the European. The absence of arbitrage implies a relationship between the price of a European put option having exercise price K and expiration time t and the price of a call option on that stock that also has exercise price K and expiration time t. This is known as the put-call option parity formula, which may be stated as follows.

Proposition 4.2.2:

Let C be the price of a call option that enables its holder to buy one share of a stock at an exercise price K at time t; also, let P be the price of a European put option that enables its holder to sell one share of the stock for the amount K at time t. Let S be the price of the stock at time 0 . Then, assuming that interest is continuously discounted at a nominal rate r, either

$S+P-C=Ke^{-rt}$ or there is an arbitrage opportunity.

Proof:

If $S+P-C<Ke^{-rt}$

then we can effect a sure win by initially buying one share of the stock, buying one put option, and selling one call option. This initial payout of $S+P-C$ is borrowed from a bank to be repaid at time t. Let us now consider the value of our holdings at time t. There are two cases that depend on $S(t)$, the stock's market price at time t. If $S(t)\leq K$ then
the call option we sold is worthless, and we can exercise our put option to sell the stock for the amount K. On the other hand, if $S(t)>K$ then our put option is worthless and the call option we sold will be exercised, forcing us to sell our stock for the price K. Thus, in either case we will

realize the amount K at time t. Since $K > e^{rt}(S + P - C)$, we can pay off our bank loan and realize a positive profit in all cases. When $S + P - C > Ke^{-rt}$,

we can make a sure profit by reversing the procedure just described. Namely, we now sell one share of stock, sell one put option, and buy one call option. We leave the details of the verification to the reader.

The arbitrage principle also determines the relationship between the present price of a stock and the contracted price to buy the stock at a specified time in the future. Our next two examples are related to these forwards contracts.

Example 4.2.3:

Forwards Contracts Let S be the present market price of a specified stock. In a forwards agreement, one agrees at time 0 to pay the amount F at time t for one share of the stock that will be delivered at the time of payment. That is, one contracts a price for the stock, which is to be delivered and paid for at time t. We will now present an arbitrage argument to show that, if interest is continuously discounted at the nominal interest rate r, then in order for there to be no arbitrage opportunity we must have

$$F = Se^{rt}$$

To see why this equality must hold, suppose that instead

$$F < Se^{rt}$$

In this case, a sure win is obtained by selling the stock at time 0 with the understanding that you will buy it back at time t. Put the sale proceeds S into a bond that matures at time t and, in addition, buy a forwards contract for delivery of one share of the stock at time t. Thus, at time t you will receive Se^{rt} from your bond. From this, you pay F to obtain one share of the stock, which you then return to settle your obligation. You thus end with a positive profit of $Se^{rt} - F$. On the other hand, if

$F > Se^{rt}$

then you can guarantee a profit of $F - Se^{rt}$ by simultaneously selling a forwards contract and borrowing S to purchase the stock.

At time t you will receive F for your stock, out of which you repay your loan amount of Se^{rt}.

When one purchases a share of a stock in the stock market, one is purchasing a share of ownership in the entity that issues the stock. On the other hand, the commodity market deals with more concrete objects: agricultural items like oats, corn, or wheat; energy products like crude oil and natural gas; metals such as gold, silver, or platinum; animal parts such as hogs, pork-bellies, and beef; and so on.

Almost all of the activity on the commodities market is involved with contracts for future purchases and sales of the commodity.

Thus, for instance, you could purchase a contract to buy natural gas in 90 days for a price that is specified today.

(Such a futures contract differs from a forward's contract in that, although one pays in full when delivery is taken for both, in futures contracts one settles up on a daily basis depending on the change of the price of the futures contract on the commodity exchange.) You could also write a futures contract that obligates you to sell gas at a specified price at a specified time.

Most people who play the commodities market never have any actual contact with the commodity. Rather, an individual who buys a futures contract most often sells that contract before the delivery date.

The relationship given in Example 4.2.2 does not hold for futures contracts in the commodity market. For one thing, if $F > Se^{rt}$ and you purchase the commodity (say, crude oil) to sell back at time t, then you will incur additional costs related to storing and insuring the oil.

Also, when $F < Se^{rt}$, selling the commodity for today's price requires that you be able to deliver it immediately.

One of the most popular types of forward contracts involves currency exchanges, the topic of our next example.

Example 4.2.4:

The September 4, 1998, edition of the New York Times gives the following listing for the price of a German mark (or DM):

- today -. 5777;
- 90-day forward -. 5808.

In other words, you can purchase 1 DM today at the price of $\$.5777$. In addition, you can sign a contract to purchase 1 DM in 90 days at a price, to be paid on delivery, of $\$.5808$. Why are these prices different?

Solution:

One might suppose that the difference is caused by the market's expectation of the worth in 90 days of the German DM relative to the U.S. dollar. However, it turns out that the entire price differential is due to the different interest rates in Germany and in the United States. Suppose that interest in both countries is continuously compounded at nominal yearly rates: r_μ in the United States and r_g in Germany. Let S denote the present price of 1 DM , and let F be the price for a forwards contract to be delivered at time t.

(This example considers the special case where $S = 0.5777, F = 0.5808$, and $t = 90/365$.)

We now argue that, in order for there not to be an arbitrage opportunity, we must have

$$F = Se^{(r_u - r_g)t}$$

suppose first that $Fe^{r_B t} > Se^{r_u t}$

To obtain a sure win, borrow S dollars to be repaid at time t.

Use these dollars to buy 1 DM , which in turn is used to buy a German bond that matures at time t. (Thus, at time t you will have $e^{r_8 t}$ German marks and you will owe an American bank $Se^{r_w t}$ dollars.)

Also, write a contract to sell $e^{r_g t}$ German marks at time t for a total price of $Fe^{r_8 t}$ dollars. Then, at time t you sell your German marks for $\mathrm{F}e^{r_8 t}$ dollars, use $Se^{r_w t}$ of this to pay off your American debt, and end with a profit of $\mathrm{F}e^{r_8 t} - Se^{r_u t}$. On the other hand, if

$Fe^{r_g t} < Se^{r_\mu t}$

then you can obtain a sure win by reversing the preceding operation as follows: At time 0 , buy a futures contract to purchase $e^{r_s t}$DM at time t; borrow 1 DM from a German bank and sell it for S dollars, which you then use to buy an American bond maturing at time t.

Thus, at time t you will have $Se^{r_u t}$ dollars; use $Fe^{r_g t}$ of it to pay for your futures contract. This gives you $e^{r_s t}$ marks, which you then use to retire your German bank loan debt. Hence, you end with a positive profit of $Se^{r_u t} - \mathrm{F}e^{r_x t}$.

Exercise:

1. Suppose it is known that the price of a certain security after one period will be one of the r values $s_1, \ldots, s_r$. What should be the cost of an option to purchase the security at time 1 for the price K when $K < \min s_i$?
2. Let C be the price of a call option to purchase a security whose present price is S. Argue that $C \leq S$.
3. Let P be the price of a put option to sell a security, whose present price is S, for the amount K. Which of the following are true?

   (a) $P \leq S$;

   (b) $P \leq K$.

4. Let P be the price of a put option to sell a security, whose present price is S, for the amount K. Argue that $P \geq Ke^{-rt} - S$,where t is the exercise time and r is the interest rate.
5. With regard to Proposition 4.2.2, verify that the strategy of selling one share of stock, selling one put option, and buying one call option is always a winning strategy when $S + P - C > Ke^{-rt}$.
6. Explain why the price of an American put option having exercise time t cannot be less than the price of a second put option on the same security that is identical to the first option except that its exercise time is earlier.
7. Is the result of the preceding exercise valid for European puts?
8. If a stock is selling for a price s immediately before it pays a dividend d (i.e., the amount d per share is paid to every shareholder), then what should its price be immediately after the dividend is paid?
9. Let $S(t)$ be the price of a given security at time t. All of the following options have exercise time t and (unless stated otherwise) exercise price K. Give the payoff at time t that is earned by an investor who:

   (a) owns one call and one put option;

   (b) owns one call having exercise price K_1 and has sold one put having exercise price K_2;

   (c) owns two calls and has sold short one share of the security;

   (d) owns one share of the security and has sold one call.

UNIT-V:

The Arbitrage Theorem

Chapter 5: Sections: 5.1-5.3

5. The Arbitrage Theorem

5.1 The Arbitrage Theorem:

Consider an experiment whose set of possible outcomes is $\{1,2,\ldots,m\}$, and suppose that n wagers concerning this experiment are available.

If the amount x is bet on wager i, then $xr_i(j)$ is received if the outcome of the experiment is

$j (j = 1,\ldots,m)$. In other words, $r_i(\cdot)$ is the return function for a unit bet on wager i.

The amount bet on a wager is allowed to be positive, negative, or zero.

A betting strategy is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, with the interpretation that x_1 is bet on wager $1, x_2$ is bet on wager $2, \ldots, x_n$ is bet on wager n.

If the outcome of the experiment is j, then the return from the betting strategy $\mathbf{x}$ is given by return from $\mathbf{x} = \sum_{i=1}^{n} x_i r_i(j)$

The following result, known as the arbitrage theorem, states that either there exists a probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ on the set of possible outcomes of the experiment under which the expected return of each wager is equal to zero, or else there exists a betting strategy that yields a positive win for each outcome of the experiment.

Theorem 5.1.1 (The Arbitrage Theorem)

Exactly one of the following is true: Either

(a) there is a probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ for which

$\sum_{j=1}^{m} p_j r_i(j) = 0$ for all $i = 1, \ldots, n$, or else

(b) there is a betting strategy $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for which $\sum_{i=1}^{n} x_i r_i(j) > 0$ for all $j = 1, \ldots, m$

Proof:

If X is the outcome of the experiment, then the arbitrage theorem states that either there is a set of probabilities $(p_1, p_2, \ldots, p_m)$ such that if

$P\{X = j\} = p_j$ for all $j = 1, \ldots, m$

$E[r_i(X)] = 0$ for all $i = 1, \ldots, n,$

Then or else there is a betting strategy that leads to a sure win. In other words, either there is a probability vector on the outcomes of the experiment that results in all bets being fair, or else there is a betting scheme that guarantees a win.

Example 5.1.1:

In some situations, the only type of wagers allowed are ones that choose one of the outcomes

$i(i = 1, \ldots, m)$ and then bet that i is the outcome of the experiment. The return from such a bet is often quoted in terms of odds. If the odds against outcome i are o_i (often expressed as " o_i to 1 "), then a one-unit bet will return either o_i if i is the outcome of the experiment or -1 if i is not the outcome. That is, a one-unit bet on i will either win o_i or lose 1. The return function for such a bet is given by

$$r_i(j) = \begin{cases} o_i & \text{if } j = i \\ -1 & \text{if } j \neq i \end{cases}$$

Suppose that the odds $o_1, o_2, \ldots, o_m$ are quoted. In order for there not to be a sure win, there must be a probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ such that, for each $i(i = 1, \ldots, m)$,

$$0 = E_{\mathrm{p}}[r_i(X)] = o_i p_i - (1 - p_i).$$

That is, we must have

$$p_i = \frac{1}{1 + o_i}.$$

Since the p_i must sum to 1 , this means that the condition for there not to be an arbitrage is that

$$\sum_{i=1}^{m} \frac{1}{1 + o_i} = 1$$

That is, if $\sum_{i=1}^{m} (1 + o_i)^{-1} \neq 1$, then a sure win is possible. For instance, suppose there are three possible outcomes and the quoted odds are as follows.

Outcome	Odds
1	1
2	2
3	3

That is, the odds against outcome 1 are 1 to 1; they are 2 to 1 against outcome 2; and they are 3 to 1 against outcome 3. Since

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} \neq 1$$

a sure win is possible. One possibility is to bet -1 on outcome 1 (so you either win 1 if the outcome is not 1 or you lose 1 if the outcome is 1) and bet -0.7 on outcome 2 (so you either win 0.7 if the outcome is not 2 or you lose 1.4 if it is 2), and -0.5 on outcome 3 (so you either win 0.5 if the outcome is not 3 or you lose 1.5 if it is 3).

If the experiment results in outcome 1 , you win $-1+0.7+.5=0.2$; if it results in outcome 2 , you win $1-1.4+0.5=0.1$; if it results in outcome 3 , you win $1+0.7-1.5=0.2$. Hence, in all cases you win a positive amount.

Example 5.1.2:

Let us reconsider the option pricing example of Section 5.1, where the initial price of a stock is 100 and the price after one period is assumed to be either 200 or 50. At a cost of c per share, we can purchase at time 0 the option to buy the stock at time 1 for the price of 150 . For what value of c is no sure win possible?

Solution:

In the context of this section, the outcome of the experiment is the value of the stock at time 1; thus, there are two possible outcomes.

There are also two different wagers: to buy (or sell) the stock, and to buy (or sell) the option. By the arbitrage theorem, there will be no sure win if there are probabilities $(p, 1-p)$ on the outcomes that make the expected present value return equal to zero for both wagers.

The present value return from purchasing one share of the stock is

$$\text{return } = \begin{cases} 200(1+r)^{-1}-100 & \text{if the price is } 200 \text{ at time } 1 \\ 50(1+r)^{-1}-100 & \text{if the price is } 50 \text{ at time } 1 \end{cases}$$

Hence, if p is the probability that the price is 200 at time 1 , then

$$\begin{aligned} E[\text{ return }] &= p\left[\frac{200}{1+r}-100\right]+(1-p)\left[\frac{50}{1+r}-100\right] \\ &= p\frac{150}{1+r}+\frac{50}{1+r}-100 \end{aligned}$$

Setting this equal to zero yields that

$$p = \frac{1+2r}{3}$$

Therefore, the only probability vector $(p, 1-p)$ that results in a zero expected return for the wager of purchasing the stock has $p = (1+2r)/3$.

In addition, the present value return from purchasing one option is

$$\text{return} = \begin{cases} 50(1+r)^{-1} - c & \text{if the price is } 200 \text{ at time } 1 \\ -c & \text{if the price is } 50 \text{ at time } 1 \end{cases}$$

Hence, when $p = (1+2r)/3$, the expected return of purchasing one option is

$$E[\text{ return }] = \frac{1+2r}{3}\frac{50}{1+r} - c$$

It thus follows from the arbitrage theorem that the only value of c for which there will not be a sure win is

$$c = \frac{1+2r}{3}\frac{50}{1+r}$$

that is, when

$$c = \frac{50+100r}{3(1+r)}$$

which is in accord with the result of Section 5.1.

5.2. The Multi period Binomial Model:

Let us now consider a stock option scenario in which there are n periods and where the nominal interest rate is r per period. Let $S(0)$ be the initial price of the stock, and for $i = 1, \ldots, n$ let $S(i)$ be its price at i time periods later.

Suppose that $S(i)$ is either $uS(i-1)$ or $dS(i-1)$, where $d < 1 + r < u$. That is, going from one time period to the next, the price either goes up by the factor u or down by the factor d. Furthermore, suppose that at time 0 an option may be purchased that enables one to buy the stock after n periods have passed for the amount K.

In addition, the stock may be purchased and sold anytime within these n time periods.
Let X_i equal 1 if the stock's price goes up by the factor u from period $i-1$ to i, and let it equal 0 if that price goes down by the factor d.

That is, $X_i = \begin{cases} 1 & \text{if } S(i) = uS(i-1) \\ 0 & \text{if } S(i) = dS(i-1) \end{cases}$

The outcome of the experiment can now be regarded as the value of the vector $(X_1, X_2, \ldots, X_n)$. It follows from the arbitrage theorem that, in order for there not to be an arbitrage opportunity, there must be probabilities on these outcomes that make all bets fair. That is, there must be a set of probabilities

$P\{X_1 = x_1, \ldots, X_n = x_n\}, x_i = 0,1, i = 1, \ldots, n$ that make all bets fair. Now consider the following type of bet: First choose a value of $i(i = 1, \ldots, n)$ and a vector $(x_1, \ldots, x_{i-1})$ of zeros and ones, and then observe the first $i-1$ changes. If $X_j = x_j$ for each

$j = 1, \ldots, i-1$, immediately buy one unit of stock and then sell it back the next period. If the stock is purchased, then its cost at time $i-1$ is $S(i-1)$; the time- $(i-1)$ value of the amount obtained when it is then sold at time i is either

$(1+r)^{-1}uS(i-1)$ if the stock goes up or $(1+r)^{-1}dS(i-1)$ if it goes down.

Therefore, if we let $\alpha = P\{X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\}$

denote the probability that the stock is purchased, and let

$p = P\{X_i = 1 \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\}$

denote the probability that a purchased stock goes up the next period, then the expected gain on this bet (in time-($i-1$) units) is

$$\alpha[p(1+r)^{-1}uS(i-1)+(1-p)(1+r)^{-1}dS(i-1)-S(i-1)].$$

Consequently, the expected gain on this bet will be zero, provided that

$$\frac{pu}{1+r}+\frac{(1-p)d}{1+r}=1$$

or, equivalently, that

$$p=\frac{1+r-d}{u-d}$$

In other words, the only probability vector that results in an expected gain of zero for this type of bet has

$$P\{X_i=1 \mid X_1=x_1,\dots,X_{i-1}=x_{i-1}\}=\frac{1+r-d}{u-d}$$

Since $x_1,\dots,x_n$ are arbitrary, this implies that the only probability vector on the set of outcomes that results in all these bets being fair is the one that takes $X_1,\dots,X_n$ to be independent random variables with

$P\{X_i=1\}=p=1-P\{X_i=0\}, i=1,\dots,n,$ ………….. (5.1)

Where $p=\frac{1+r-d}{u-d}$ …………..(5.2)

It can be shown that, with these probabilities, any bet on buying stock will have zero expected gain. Thus, it follows from the arbitrage theorem that either the cost of the option must be equal to the expectation of the present (i.e., the time-0) value of owning it using the preceding probabilities, or else there will be an arbitrage opportunity.

So, to determine the no-arbitrage cost, assume that the X_i are independent 0 -or-1 random variables whose common probability p of being equal to 1 is given by Equation (5.2). Letting Y denote their sum, it follows that Y is just the number of the X_i that are equal to 1 , and thus Y is a binomial random variable with parameters n and p. Now, in going from period to period, the stock's price is its old price multiplied by either u or by d. At time n, the price would have gone up Y times and down $n - Y$ times, so it follows that the stock's price after n periods can be expressed as

$$S(n) = u^Y d^{n-Y} S(0)$$

where $Y = \sum_{i=1}^{n} X_i$ is, as previously noted, a binomial random variable with parameters n and p. The value of owning the option after n periods have elapsed is $(S_n - K)^+$, which is defined to equal either $S_n - K$ (when this quantity is nonnegative) or zero (when it is negative). Therefore, the present (time-0) value of owning the option is

$$(1+r)^{-n}(S(n) - K)^+$$

and so the expectation of the present value of owning the option is

$$(1+r)^{-n}E[(S(n) - K)^+] = (1+r)^{-n}E[(S(0)u^Y d^{n-Y} - K)^+]$$

Thus, the only option cost C that does not result in an arbitrage is

$$C = (1+r)^{-n}E[(S(0)u^Y d^{n-Y} - K)^+] \quad \text{............ (5.3)}$$

Remark:

Although Equation (6.3) could be streamlined for computational convenience, the expression as given is sufficient for our main purpose: determining the unique no-arbitrage option cost when the underlying security follows a geometric Brownian motion. This is accomplished in our next chapter, where we derive the famous Black-Scholes formula.

5.3. Proof of the Arbitrage Theorem:

In order to prove the arbitrage theorem, we first present the duality theorem of linear programming as follows. Suppose that, for given constants c_i, b_j, and

$a_{i,j}(i = 1, \dots, n, j = 1, \dots, m)$, we want to choose values $x_1, \dots, x_n$ that will

$$\text{maximize} \sum_{i=1}^{n} c_i x_i$$

subject to

$$\sum_{i=1}^{n} a_{i,j} x_i \leq b_j, j = 1,2, \dots, m.$$

This problem is called a primal linear program. Every primal linear program has a dual problem, and the dual of the preceding linear program is to choose values $y_1, \dots, y_m$ that

$$\text{minimize} \sum_{j=1}^{m} b_j y_j$$

subject to

$$\sum_{j=1}^{m} a_{i,j} y_j = c_i, i = 1, \dots, n$$

$$y_j \geq 0, j = 1, \dots, m$$

A linear program is said to be feasible if there are variables $(x_1, \dots, x_n$ in the primal linear program or $y_1, \dots, y_m$ in the dual) that satisfy the constraints.

Proposition 5.3.1

(Duality Theorem of Linear Programming) If a primal and its dual linear program are both feasible, then they both have optimal solutions and the maximal value of the primal is equal to the minimal value of the dual.

If either problem is infeasible, then the other does not have an optimal solution.

A consequence of the duality theorem is the arbitrage theorem. Recall that the arbitrage theorem refers to a situation in which there are n wagers with payoffs that are determined by the result of an experiment having possible outcomes $1,2,\ldots,m$.

Specifically, if you bet wager i at level x, then you win the amount $xr_i(j)$ if the outcome of the experiment is j.

A betting strategy is a vector $\mathbf{x} = (x_1,\ldots,x_n)$, where each x_i can be positive or negative (or zero), and with the interpretation that you simultaneously bet wager i at level x_i for each $i = 1,\ldots,n$.

If the outcome of the experiment is j, then your winnings from the betting strategy $\mathbf{x}$ are

$$\sum_{i=1}^{n} x_i r_i(j)$$

Proposition 5.3.2 (Arbitrage Theorem):

Exactly one of the following is true: Either
(i) there exists a probability vector $\mathbf{p} = (p_1,\ldots,p_m)$ for which

$$\sum_{j=1}^{m} p_j r_i(j) = 0 \quad \text{for all } i = 1,\ldots,n$$

(or)

(ii) there exists a betting strategy $\mathbf{x} = (x_1,\ldots,x_n)$ such that

$$\sum_{i=1}^{n} x_i r_i(j) > 0 \quad \text{for all } j = 1,\ldots,m$$

That is, either there exists a probability vector under which all wagers have expected gain equal to zero, or else there is a betting strategy that always results in a positive win.

Proof:

Let x_{n+1} denote an amount that the gambler can be sure of winning, and consider the problem of maximizing this amount. If the gambler uses the betting strategy $(x_1, \ldots, x_n)$ then she will win $\sum_{i=1}^{n} x_i r_i(j)$ if the outcome of the experiment is j.

Hence, she will want to choose her betting strategy $(x_1, \ldots, x_n)$ and x_{n+1} so as to maximize x_{n+1}
subject to

$$\sum_{i=1}^{n} x_i r_i(j) \geq x_{n+1}, j = 1, \ldots, m$$

$$a_{i,j} = -r_i(j), i = 1, \ldots, n, a_{n+1,j} = 1$$

we can rewrite the preceding as follows:

maximize x_{n+1}
subject to

$$\sum_{i=1}^{n+1} a_{i,j} x_i \leq 0, j = 1, \ldots, m$$

Note that the preceding linear program has $c_1 = c_2 = \cdots = c_n = 0$, $c_{n+1} = 1$, and upper-bound constraint values all equal to zero (i.e., all $b_j = 0$). Consequently, its dual program is to choose variables $y_1, \ldots, y_m$ so as to

minimize 0
subject to

$$\sum_{j=1}^{m} a_{i,j} y_j = 0, i = 1, \ldots, n$$

$$\sum_{j=1}^{m} a_{n+1,j} y_j = 1$$

$$y_j \geq 0, j = 1, \ldots, m$$

Using the definitions of the quantities $a_{i,j}$ gives that this dual linear program can be written as

minimize 0

subject to

$$\sum_{j=1}^{m} r_i(j)y_j = 0, i = 1, \ldots, n$$

$$\sum_{j=1}^{m} y_j = 1$$

$$y_j \geq 0, j = 1, \ldots, m$$

Observe that this dual will be feasible, and its minimal value will be zero, if and only if there is a probability vector $(y_1, \ldots, y_m)$ under which all wagers have expected return 0 .

The primal problem is feasible because $x_i = 0(i = 1, \ldots, n + 1)$ satisfies its constraints, so it follows from the duality theorem that if the dual problem is also feasible then the optimal value of the primal is zero and hence no sure win is possible.

On the other hand, if the dual is infeasible then it follows from the duality theorem that there is no optimal solution of the primal. But this implies that zero is not the optimal solution, and thus there is a betting scheme whose minimal return is positive.

(The reason there is no primal optimal solution when the dual is infeasible is because the primal is unbounded in this case. That is, if there is a betting scheme x that gives a guaranteed return of at least $v > 0$, then $c\mathbf{x}$ gives a guaranteed return of at least cv.)

Exercises:

1.Consider an experiment with three possible outcomes and odds as follows.

Outcome	Odds
1	1
2	2
3	5

Is there a betting scheme that results in a sure win?

2.Consider an experiment with four possible outcomes, and suppose that the quoted odds for the first three of these outcomes are as follows.

Outcome	Odds
1	2
2	3
3	4

What must be the odds against outcome 4 if there is to be no possible arbitrage when one is allowed to bet both for and against any of the outcomes?

3. In Example 5.1.1, show that if $\sum_{i=1}^{m} \frac{1}{1+o_i} \neq 1$

then the betting scheme

$$x_i = \frac{(1+o_i)^{-1}}{1-\sum_{i=1}^{m} (1-o_i)^{-1}}, i = 1, \dots, m$$

will always yield a gain of exactly 1.

4.In Example 5.1.2, suppose one also has the option of purchasing a put option that allows its holder to put the stock for sale at the end of one period for a price of 150. Determine the value of P, the cost of the put, if there is to be no arbitrage; then show that the resulting call and put prices satisfy the put-call option parity formula (Proposition 5.2.2).

5.Suppose that, in each period, the cost of a security either goes up by a factor of 2 or down by a factor of $1/2$ (i.e., $u = 2, d = 1/2$). If the initial price of the security is 100, determine the no-arbitrage cost of a call option to purchase the security at the end of two periods for a price of 150

6. Suppose, in Example 5.1.2, that there are three possible prices for the security at time 1: 50, 100 , or 200 . (That is, allow for the possibility that the security's price remains unchanged.) Use the arbitrage theorem to find an interval for which there is no arbitrage if c lies in that interval.

A betting strategy **x** such that (using the notation of Section 5.1)

$$\sum_{i=1}^{n} x_i r_i(j) \geq 0, i = 1, \ldots, m$$

with strict inequality for at least one i, is said to be a weak arbitrage strategy. That is, whereas an arbitrage is present if there is a strategy that results in a positive gain for every outcome, a weak arbitrage is present if there is a strategy that never results in a loss and results in a positive gain for at least one outcome. (An arbitrage can be thought of as a free lunch, whereas a weak arbitrage is a free lottery ticket.) It can be shown that there will be no weak arbitrage if and only if there is a probability vector **p**, all of whose components are positive, such that

$$\sum_{j=1}^{m} p_j r_i(j) = 0, i = 1, \ldots, n$$

In other words, there will be no weak arbitrage if there is a probability vector that gives positive weight to each possible outcome and makes all bets fair.

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